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ABSTRACT

The theme of the sixth annual meeting of the North American Chapter of Psychology of Mathematics Education was "The Importance of Psychological Models for Research in Mathematics Learning." The conference proceedings contain papers organized around five major topics. In the first section, eight papers on the learning of arithmetic are included. The second section contains six papers on the learning of algebra. The learning of geometry and measurement is also the focus of six papers. In the fourth section, eleven papers on problem solving are included. The fifth section, on classroom environment, contains six papers, including two on conceptions of mathematics and two on computer-based instruction. Finally, abstracts of six symposia are included. (MNS)

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PROCEEDINGS OF THE
SIXTH ANNUAL MEETING

PME-NA

NORTH AMERICAN CHAPTER OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

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3-6 October 1984

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PROCEEDINGS OF THE
SIXTH ANNUAL MEETING

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NORTH AMERICAN CHAPTER OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

Edited by
JAMES M. MOSER

MADISON, WISCONSIN
3-6 October 1984

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PREFACE

The International Group for the Psychology of Mathematics Education (PME) was founded in 1976 at the Third International Congress for Mathematics Education held in Karlsruhe, West Germany. The North American Chapter (PME-NA) came to life three years later in 1979 with its first meeting in Evanston on the campus of Northwestern University. Both groups sponsor an annual research conference bringing together mathematics educators, psychologists, mathematicians, teacher educators, and more recently computer scientists, all interested in the general problems of teaching and learning mathematics. A joint meeting of the two groups was held in Berkeley, California, immediately following the fourth International Congress for Mathematics Education. Since then, PME-NA has met in Minneapolis/St. Paul, Athens, and Montreal and, now for the Sixth Annual Meeting on the campus of the University of Wisconsin at Madison. We, the organizers, and we trust the members and participants as well, take great pride in the fact that PME-NA is truly becoming north american by reason of the greatly increased participation of Mexican colleagues this year as well as the continued participation and support of Canadian and American (read: "USA") researchers. Obviously, the shared collegiality of mathematics educators from the major north american countries of Mexico, Canada, and the USA makes us truly international as well, but we also wish to acknowledge the contributions to be made by participants from Europe, Israel, and Africa. We sincerely hope that this will be just the beginning of increased sharing of ideas among persons from many nations.

Following the idea advanced last year at the meeting in Montreal, we have chosen a theme for this year's conference. It is "The Importance of Psychological Models for Research in Mathematics Learning." Several invited speakers will present their view on the subject and a panel discussion with reactions will follow. It is hoped that all participants

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will keep this theme in mind as they present their own work and as they react and discuss the work of others.

In an attempt to return to more lively and in-depth discussion of issues, research, and methodology, we have instituted a series of symposia into the program. Certainly, the notion of a symposium is not a new or startling idea, but they have not been used to any degree in previous PME-NA meetings. We, the organizers, had the fear that PME-NA was running the risk of simply becoming a forum for people to present their research findings or theories without providing sufficient time for interaction and discussion and debate and friendly, constructive criticism. We hope that the program will allow more time for such scholarly activity.

Another feature of the program that must be mentioned but which is not at all evident from the printed proceedings or schedule of events is the fact that proposals for presentations and symposia were peer-reviewed by members of the Steering Committee listed on page v and by the editor. We felt that this all-important aspect had to be more rigorously pursued than it had been during the early growing years of PME-NA. It is to the benefit of all that this process be continued in future years so that PME-NA can continue to grow as a leader in its field.

The editor wishes to gratefully acknowledge the assistance given to him by the PME-NA Steering Committee and particularly to his colleagues, Tom Carpenter and Tom Romberg. A very special thanks is due to the cheerful assistance and valuable service provided by my secretary, Dorothy Egener. Finally, we once again wish to mention our sincere appreciation to the moral and financial assistance provided by the Wisconsin Center for Education Research and its director, Mike Smith.

Jim Moser
Madison, Wisconsin
October 1984

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TABLE OF CONTENTS

	Page
THE LEARNING OF ARITHMETIC	1
The Case of Brian: An Additional Explanation for Production Deficiencies *Arthur J. Baroody and Cathleen A. Mason*	2
Introducing Subtraction As Counting Up *Karen C. Fuson and James W. Hall*	9
First Graders' Solution Strategies of Addition and Subtraction Word Problems *Erik De Corte and Lieven Verschaffel*	15
Researchers Taking in Charge a Class: Results of a Three Year Project on Numeration *Bernadette Janvier*	21
Understanding Multiplication and Division *Brian Greer and Clare Mangan*	27
Tasks to Measure the Development of Computational Estimation Skills and Concepts *Judith Threadgill-Sowder and Margarieta Montague Wheeler*	33
The Use of Operator Functions to Teach Rational Number Concepts *Frederick O. Flener and James R. Lockwood*	37
Mental Computation Proficiency, Errors and Strategies of 7th and 8th Grade Students on Two Operations with Decimal Fractions *Choshi D. Kasanda*	43
THE LEARNING OF ALGEBRA	49
From an Arithmetical to an Algebraic Thought (A Clinical Study with 12-13 Year Olds) *Eugenio Filloy and Teresa Rojano*	51
Constructing the Notion of Variable Using History and Bottles *Claude Janvier*	57

TABLE OF CONTENTS (Continued)

	Page
Using Literal Symbols to Represent Hidden Quantities *Nicolas Herscovics and Louise Chalouh*	64
From Letter Representing a Hidden Quantity to Letter Representing an Unknown Quantity *Louise Chalouh and Nicolas Herscovics*	71
The Role of Cognitive Model Building: A Case Study *James J. Kaput*	77
A Comparison between Novice and More-expert Algebra Students on Tasks Dealing with the Equivalence of Equations *Carolyn Kieran*	83
THE LEARNING OF GEOMETRY AND MEASUREMENT	93
A First Approach to Measuring (Children between 11-13 Years Old) *Olimpia Figueras and Guillermina Waldegg*	95
The Communication of Spatial Information by Means of Coded Orthogonal Views *Claude Gaulin, Gerald Noelting, and Eva Puchalska*	101
Interior Structurings of Geometrical Objects: An Example of Infralogical Groupings *Dieter Lunkenbein*	107
Van Hiele Levels of Thinking of Sixth Graders *David J. Fuys*	113
The Atlanta - Emory Logo Project: Teaching and Understanding Geometric Relationships through Logo *John Olive and Susan Pauls Scally*	120
The Growth of Similarity Concepts at the Sixth Grade Level *Alex Friedlander, Willian M. Fitzgerald, and Glenda Lappan*	127

TABLE OF CONTENTS (Continued)

	Page
PROBLEM SOLVING	133
The Importance of Beliefs and Expectations in the Problem Solving Performance of Second Grade Pupils *Paul Cobb*	135
The Importance of Beliefs and Expectations in the Problem Solving Performance of Sixth Grade Pupils *Grayson H. Wheatley*	141
Mathematical Problem-solving Processes of Average-ability Middle-grade Students Working in Small Groups *Lynn C. Hart*	147
Self-directed Use of Different Types of Representational Models in Middle School Problem Solving *Karen Schultz*	153
Links among Conceptual Understanding, Computational Skill, and Applied Problem Solving Ability *Richard Lesh*	159
Barriers to Mathematical Thinking and Problem Solving *Donald N. Peck*	165
The Effects of Spatial Ability and Problem Presentation Format on Mathematical Problem Solving Performance of Middle School Students *Marsha Landau*	171
Selection of Mathematical Problem-solving Heuristics: A Research Proposal *Joe Garofalo and Frank K. Lester, Jr.*	178
The Structure of Heuristic Processes *Gerald A. Goldin*	183
Concept-driven Solutions to Routine Story Problems *Larry Souder*	189
Basic Problem Solving Skills As Prerequisites for Advanced Problem Solving Skills in Mathematics and Science *John Clement*	194

TABLE OF CONTENTS (Continued)

	Page
CLASSROOM ENVIRONMENT	201
Research on the Role of Play and Games in the Learning of Mathematics *Claudette Larouche, Jacques C. Bergeron and Nicolas Herscovics*	203
Creating Classroom Environments Effective for Learning Mathematics *Carolyn A. Maher*	213
The Perception of Mathematics among Elementary School Teachers *Jean J. Dionne*	223
The Skills-understanding Dilemma in Mathematics Education *Nicole Nantais, Nicolas Herscovics, and Jacques C. Bergeron*	229
How to Use a Computer to Teach Mathematical Concepts *Tommy Dreyfus*	239
Effect of Computer Graphics on Improving Estimates to Average Speed Problems *Stephen K. Reed and Holly Saavedra*	245
SYMPOSIUM	251
From Counting to Recall of Number Facts *Ruth Steinberg*	253
Affective Factors and Mathematics Learning *Elisabeth Fennema*	263
Sex-related Differences in Mathematics Learning *Mary Sohata Koehler*	265
The Role of Computer Programming in the Development of Mathematical Concepts *Patricia Wilson*	266

TABLE OF CONTENTS (Continued)

	Page
Symposium Proposal *Merlyn J. Behr*	269
The Teaching and Learning of Geometry *Alan H. Schoenfeld*	270
A Constructivist Approach to Mathematics Instruction *Jere Confrey*	272
Author Index with Mailing Addresses	275

THE LEARNING OF ARITHMETIC

THE CASE OF BRIAN: AN ADDITIONAL EXPLANATION FOR PRODUCTION DEFICIENCIES

Arthur J. Baroody and Cathleen A. Mason, University of Rochester

Abstract

Counting out (production of) a subset from a larger set is a relatively difficult counting task. Production deficiencies such as counting beyond the requested amount have been attributed to memory difficulties. A case study with a mentally retarded child was undertaken to examine the possibility that production difficulties may be due to a conceptual deficit rather than a memory problem. The study found that, though the child could remember the requested amount, he did not seem to appreciate the cardinal-count transition (that the cardinal designation of the requested amount predicted or was equivalent to counting a set of five).

This research was supported by NICHD grant HD 16757-01 and HD 16757-02.

Counting out (producing) a specified number of objects is a relatively difficult counting task because of the multiple demands put on working memory: (a) register the target (the cardinal term requested); (b) label each object taken with a number tag from the natural numbers sequence; and (c) compare the number tags against the target in order to stop the counting process. A common error among young and mentally retarded children is a failure to stop the counting process after reaching the target (Baroody & Snyder, 1983; Spradlin, Cotter, Stevens, & Friedman, 1974; Wang, Resnick & Boozer, 1971). "No-stop errors" have been attributed to working memory process failures. An additional explanation for production deficiencies is suggested by Fuson's work (e.g., Fuson & Hall, 1983) and another systematic production error recently identified in mentally retarded children (Baroody & Ginsburg, 1984). This paper reports on a case study that investigated the newly identified production error and the additional explanation for production deficiencies.

No-stop errors have been attributed to two types of memory failures (see, e.g., Resnick & Ford, 1981). According to the "register-deficit" hypothesis, children may fail to register the specified amount or, because they become so involved in the counting process, they forget the target value. In short, the child does not have a basis for stopping the production process, because s/he does not have or hold the target in working memory. According to the "matching-failure" hypothesis, a work overload prevents working memory from simultaneously processing the required information. That is, even though the child registers (and can later recall) the target (N), the counting process so taxes working memory that the child fails to match the specified N to the N in the tagging process. Thus, according to the register-deficit and matching-failure hypotheses, no stop errors are due to either not having or not keeping the goal of the task in mind.

A third possibility is that such problems are not due to a memory failure but to a conceptual deficit. Specifically, production deficiencies may be the result of an inability to make what Fuson and Hall (1983) call the "cardinal-count transition" (an ability to appreciate that, for example, the cardinal term "5" predicts or is equivalent to the outcome of counting a set of five objects). That is, some children may not realize that they are supposed to match the target N to their count.

A conceptual deficit hypothesis is consistent with a type of production error recently noticed in a study of mentally retarded children (Baroody & Ginsburg, 1984). In addition to the no-stop error made by many subjects, a few retarded children made what might be termed an "end-with-N" error. One boy, for example, was asked to count out 3 objects from a pile of 5 items and responded: "1, 3, 4, 6, 11" and then retagged the last item "3." Asked to produce 5 items, the child responded: "1, 2, 3, 4, 5, 6, 8, 9, 5." In brief, this child typically exhibited a no-stop error but frequently ended his count with the correct tag. Though the child failed to stop when the target tag was first produced, he frequently appeared to remember the goal and so made sure that the last item was given the target tag. Another TMR boy (Brian) regularly made this end-with-N error but usually after abbreviating his count. For example, in response to a request to count out four items, he responded: "1, 2, 4," or twelve items: "1, 2, 12." Again it appeared this child kept the goal in mind. It may be that the mentally retarded children described in the examples above did not have a very sophisticated concept of cardinality—that is, they could not make the cardinal-count transition. Thus, though these children remembered the specified amount, they may not have had a basis for even attempting a match.

Note that such end-with-N errors cannot be reconciled with a register-deficit hypothesis. End-with-N errors, however, are not inconsistent with a matching-failure hypothesis. In the first example, the child may have failed to make the match because he could not simultaneously count objects and match. At the end of the counting process—freed of executing two processes simultaneously—the child focused on the goal of the task and so labeled the last item with the target tag. In the second example, the child may have dealt with the overload on working memory posed by counting objects and matching by abbreviating the counting process (rather than losing track of the goal).

Two case studies were undertaken to check the plausibility of the conceptual deficit hypothesis. One child who could and a second who could not produce objects were studied. This report focuses on the second child, who consistently made an end-with-N error and who could not produce even small sets of 2 or 3. The latter criterion excluded children whose end-with-N errors are simply the result of a performance failure (a Type II error). Children who can produce small sets but who make end-with-

N errors with larger sets may well understand the task but may be simply saving themselves the labor of completing the production process.

Method

Subjects

Based on his test results from a previous study (Baroody & Ginsburg, 1984), Brian's production ability appeared to make him an ideal candidate for the case study. Except for an occasional no-stop error or problem with forgetting the target, Brian typically made an end-with-N error—even for small target Ns (e.g., for produce $3, he responded: "1, 2, 3, 4, 3"). Because he was unable to produce even small sets, which would have required little or no more effort than his faulty procedures, it did not appear that Brian's end-with-N strategy was simply a device for avoiding work. Brian, a Down's Syndrome child, was 10 years and 9 months old at the time of the case study. He scored 45 (trainable range) on an IQ test (Standard-Binet Intelligence Scale, Form L-M). The test was administered on 1/83 when Brian was 9-7, thus indicating that his mental age was 4-3 at that time. School records revealed no known medical disorders that might interfere with learning.

In order to have a basis of comparison, a second child who was clearly competent on the production task was also studied. Previous testing with Tracey (Baroody & Ginsburg, 1984) indicated her production ability contrasted sharply with that of Brian. Tracey, a Down's Syndrome child, was 10 years and 5 months at the time the case study started. She scored 58 (low educable range) on an IQ test (Stanford-Binet Intelligence Scale, Form L-M). The test was administered on 4/80 when Tracey was 6-8, thus indicating that her mental age was 4-1 at the time of testing.

Procedure

The second author individually interviewed Brian on three occasions. The interviews were videotaped and later transcribed. In the first session (February 16, 1984), two production tasks and a cardinal-count transition task were administered. Production Task 1A required the child to count out small subsets (2 to 5 items) or a larger subset (6 to 10 items) from a very large set. Production Task 2 required the child to count out small and larger subsets from a set of limited size (a task somewhat more likely than Task 1 to elicit a no-stop error). The cardinal-count transition, a somewhat more

sophisticated cardinal notion than the cardinality rule (e.g., Fuson & Hall, 1983), was gauged by two tasks. For Cardinal-Count Task 1, the tester specified the value of a set and then asked the child to predict what the outcome would be if the set were counted. Cardinal-Count Task 2 was administered in later sessions and is described below. In the Session 2 (March 3, 1984), Production Task 1B (a variation on Task 1A) and Cardinal-Count Task 2 were administered. For Cardinal-Count Task 2, the tester specified a target N and a muppet counted out the objects. The child was required to stop the muppet when it had reached the target N. That is, the child was only required to remember the target and to stop the count sequence at a specific cardinal value. In Session 3 (March 10, 1984), Production Tasks 1B and 2, and Cardinal-Count Task 2 were readministered.

Predicted Results. If the only problem underlying Brian's production deficiency was a memory failure, the boy should not recall the target N (as predicted by the register-deficit hypothesis) or be successful on just the cardinal-count task (as predicted by the matching-failure hypothesis). Note that the cardinal-count tasks should be easier than the production task because they free the child from the simultaneous demands required of working memory by the production task.

As expected, the individual interviews with Tracey indicated that she could consistently produce small sets accurately and only occasionally made a slip in producing larger sets. More importantly, she automatically responded to Cardinal-Count Task 1 with the cardinal designation of the set. With Cardinal-Count Task 2, she consistently and emphatically stopped the tester's count at the designated cardinal term. In brief, it appeared that Tracey, who was capable at the production task, was quite successful on the cardinal-count tasks, as would be predicted.

Results and Discussion

Consistent with his previous test results, Brian--at least for the first two sessions--regularly abbreviated his counting and made the end-with-N error. Brian's responses to the last two trials of Production Task 2 of Session 1 are especially interesting. In each case, Brian paused at a critical point in the count and looked at the tester as though he were seeking guidance. In one case he proceeded to produce the correct amount but, in the second, he proceeded to make a no-stop error. In this latter case,

it was clear he remembered the target: He paused after counting out 5 objects and, in response to the check question, he correctly indicated that he was supposed to have gotten 5 items. (This is clearly inconsistent with a register-deficit hypothesis.) Moreover, he appeared to make some effort at matching the target value to his count but appeared unsure whether this was what he was supposed to do. That is, he apparently made an effort to simultaneously count and match, but was uncertain whether the five tag in the sequence could represent the same quantity as the cardinal designation "five." In other words, this behavioral evidence suggests that a "conceptual deficit" played a role in Brian's production deficiency.

This interpretation is supported by Brian's performance on Cardinal-Count Tasks 1 and 2, which gauged an integral component of an ability to produce objects. It appeared that his grasp of the cardinal-count transition was not secure or automatic. Given the cardinal designation "two" and asked to predict what the outcome of counting the set would be, Brian responded by counting "1, 2"—without looking at the die. For trials involving "three" and "five," however, he resorted to counting the dots on the die. He then did make a correct prediction when the trial involving 5 was repeated. On Cardinality-Count Task 2 (Session 2), he was incorrect on the first trial, inattentive on the next two, and initially incorrect on the last trial. On the last trial, after the tester got to 8, Brian apparently realized that the target had been passed and commented: "Hey, five! 1, 2, 3, 4, 5 [counted the first five objects]—that's five!" During Session 3, he was correct on the trial involving "two," incorrect on two other trials and inattentive on a fourth trial. Note that Cardinal-Count Task 2 should be easier than the production task, because working memory is relieved of the burden of generating a count and tagging objects <u>while</u> trying to match. Nevertheless, Brian did not appear to take the cardinal-count transition for granted, and this might help to explain why his production efforts were belabored and often unsuccessful.

In conclusion, Brian's end-with-N error is clearly inconsistent with a register-deficit hypothesis. Moreover, a matching-failure hypothesis or poor test performance because of a desire to save effort were not sufficient explanations for Brian's behavior. It appears that a conceptual-deficit hypothesis must be invoked to explain, in whole or in part, Brian's production deficiency. It may be that Brian's long-standing production deficiency was due to the fact that he did not appreciate the cardinal-count transition.

That is, he did not know he had to match the N in his count to the target N (rather than forgetting to make the match). Another possibility is that until the cardinal-count transition becomes automatic, it presents an extra processing burden and thus may help overload working memory.

By logical extension, no-stop errors may be due to a conceptual deficit as well as memory failures. Indeed a conceptual deficit may help to explain why some children fail to register the target N or make a match. If the child does not appreciate the cardinal-count transition, they have no reason to register the target N or attempt a match. If the cardinal-count transition is less than automatic, the extra load on working memory may cause them to forget the target N or overlook the matching process. Clearly, further research is needed to investigate what role conceptual factors play in production deficiencies, especially among retarded children.

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INTRODUCING SUBTRACTION AS COUNTING UP

Karen C. Fuson and James W. Hall

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Five classes from Grades 1 and 2 were taught subtraction with comparison, missing addend, and take-away problems. Symbolic subtraction problems such as 14 - 8 = ? were set within these story situations, and children were taught with considerable success to solve subtraction problems by counting up from the smaller number to the larger while keeping track of the number counted up by using one-handed finger patterns. The children improved quite considerably on a timed test of the more difficult subtraction combinations, and this improvement held up over a month. They used the counting-up-to procedure with finger patterns to solve symbolic subtraction problems such as 14 - 8 = ?, and most used that approach successfully to solve comparison, missing addend, and take-away story problems.

Many studies (see Carpenter & Moser, 1984, for a recent report) indicate that children spontaneously solve subtraction problems both by counting down (e.g. for 8 - 5 = ?, "8, 7, 6, 5") and by counting up to (e.g., for 8 - 5 = ?, "5, 6, 7, 8"). Some authors have also described the considerable difficulties children display with counting down (Baroody, 1984; Fuson, 1984a; Steffe, Spikes, & Hirstein, 1976; Steinberg, 1983). In response to this situation, Fuson (1984a) suggested that subtraction might be introduced to children in ways that would lead naturally to the use of counting up to rather than counting down.

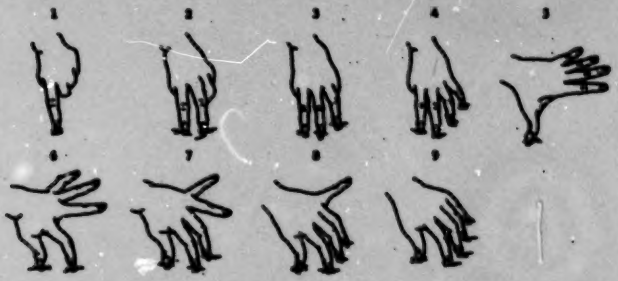
This paper reports on a study in which the recommendations of Fuson were followed. Lesson plans and student worksheets were prepared for use by five teachers of Grades 1 and 2. All of these children had been taught two months previously to count on in order to solve addition problems with sums up to 18 (see Fuson, 1984b, for a report of this project). So that children could keep track of how many words they had counted on with only one hand (and not have to put down their pencil as most children do when keeping track with both hands), the children were taught one-handed finger patterns for the numbers one through nine and were taught to use these finger patterns when counting

(see Figure 1). Counting on was initially taught within a dot and symbolic setting as in Figure 2.

Children were introduced to subtraction by being shown three kinds of subtraction story situations: comparison, take-away, and missing addend situations. Subtraction was identified as the solution procedure for the comparison, missing addend, and take-away real-world situations. The symbol for subtraction (-) was always read as "minus" and never as "take-away" so that the latter would not interfere with the understanding of comparison and missing addend problems as subtraction situations. Symbolic problems (e.g., 8 - 5 = ?) were first interpreted in comparison and missing addend story situations written and pictured on the board and were also linked to the dot and symbolic context in which the children had learned to count on (see Figure 2). Children were taught to count up to a number using finger patterns to keep track of the number of words they had counted (see Figure 1).

The subtraction teaching focused upon the most difficult subtraction problems. Each practice and test sheet had half of the problems with both parts of the whole (i.e., the known part and the unknown part) between 6 and 9 and half had one part between 6 and 9 and the other part between 3 and 5. This was done both because the goal was to teach a solution method which could be used for all subtraction problems and because the subtraction problems involving 1 and 2 are those which are most easily and most naturally solved by a counting down procedure. Baroody (1984) suggested that teaching subtraction first with small numbers may reinforce a tendency to solve subtraction problems by counting down. We wished eventually to move to having children solve -1 and -2 problems by counting down but did not want to confuse children by doing this before counting up to. We viewed the -1 and -2 subtraction problems as exceptions to the general caveats about the difficulty of counting down because in fact these are not solved by a general counting-down procedure. They are solved by saying the sequence word which comes just before the sequence word for the whole (e.g., for 8 - 1, "8, 7") or by doing that twice ("8,7,6"). These are much easier than general counting down because no keeping track is necessary and because children can produce one or two words that come before a given word more easily then they can produce words which

The finger patterns for 1 through 9 are made by touching certain fingers and/or the thumb to some surface such as a table. When there is kinaesthetic as well as visual feedback for the finger patterns. The finger patterns use a subbase of 5. The thumb is 5, and 6 is the thumb plus the 1 finger (5+1), 7 is the thumb plus the 2 fingers (5+2), etc. The motion from 4 to 5 is a very strong definite motion--the fingers all go up and the thumb goes down, all in one sharp motion with the wrist twisting.



The finger patterns shown are the patterns used in Chisenbop. We use the finger patterns differently from the way in which they are used in Chisenbop. We use them in the way that children spontaneously use fingers on both hands to keep track--the fingers just match the counting words which are continuing on to the second addend.

Addition	Subtraction
8 + 5 = ?	13 - 8 = ?
Counting on	Counting up to
Stop when finger pattern for 5 is made	Stop when say 13.
Answer is last word said.	Answer is what the hand says: The finger pattern for 5.

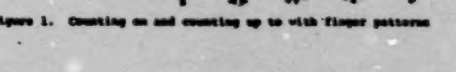


Figure 1. Counting on and counting up to with finger patterns

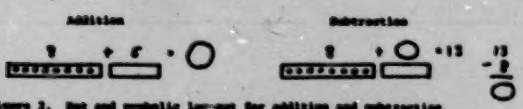


Figure 2. Dot and symbolic lay-out for addition and subtraction

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come considerably before a given word (Fuson, Richards, & Briars, 1982).

METHOD

Subjects. The subjects were 103 members of five math classrooms grouped by ability in two schools in a small heterogeneous city on the northern border of Chicago. The classes ranged from below average first graders to below average second graders.

Procedure. All students were first given a timed (two minute) subtraction pretest of 20 problems in column form (one number was written below another). Students were then given two 2-minute addition tests of 20 problems each, one of problems in column form counterbalanced with one of problems in row form. Two 2-minute subtraction posttests were given at the completion of the subtraction unit. One consisted of 20 problems in row form; the other (given in counterbalanced order) contained 20 problems in column form. These two tests were given again one month after the immediate posttest. The teaching time varied ... the five teachers and took from ten to fifteen days to complete.

RESULTS AND DISCUSSION

The children learned to implement this method of subtraction. They improved significantly from a pretest mean of 6.3 to an immediate posttest mean of 14.4 out of 20 problems completed within two minutes, $t(102) = 16.21$, $p<.01$. This level of learning was maintained over a one month period; the mean score on the one month delayed posttest was 14.7. Performance was equivalent on column and row forms of the symbolic subtraction problems. The mean scores on these two forms did not differ significantly either on the immediate posttest, 14.4 for the column form and 14.3 for the row form, or on the one month delayed posttest, 14.7 and 14.9, respectively.

The level of performance for addition and for subtraction problems was similar: the mean score on the addition test given just before the subtraction teaching began was 14.3 and the immediate posttest subtraction mean was 14.4. However, somewhat different children evidently did better on

these two types of problems, as the correlation between these two scores was only .53.

Interviews were held with the children having the five highest and the five lowest pre-post subtraction gain scores in each classroom. There were very few differences between the groups in their performance on the interview tasks. Of the 50 children, 46 spontaneously and correctly solved symbolic subtraction problems by counting up to with finger patterns. Twenty of the 50 children spontaneously and correctly used counting up to with finger patterns to solve an unfamiliar missing addend situation presented with chips; an additional 19 children correctly used counting up to with finger patterns to solve these problems when such use was suggested to them.

Most children in both groups used counting up to in order to solve all three kinds of subtraction story problems. Out of 2 problems of each kind given, the high gain children without help solved a mean of 1.7, 1.6, 1.4, and 1.3 of the comparison, missing addend, take-away, and addition story problems, respectively; a mean of 1.6, 1.6, and 1.2, respectively, were solved by counting up to. The low gain children without help solved a mean of 1.3, 1.3, 1.4, and 1.4, respectively, of these stories; a mean of 1.1, 1.2, and 1.2, respectively, were solved by counting up to. The few correct solutions that were not counting up to involved known facts, drawing sticks, fingers, and one (of the 300 subtraction solutions) counting down solution. Children demonstrated considerable competence on these problems, solving from 70% to 75% correctly. Children solved as many comparison and missing addend stories correctly as they solved take-away and addition stories.

One reason for undertaking this study was that we thought it possible that by learning at an earlier age the efficient methods of solution that later are spontaneously constructed by most children (counting on and counting up to), children might learn other mathematical concepts and procedures more easily because the child could better concentrate on the new learning. This possibility received some modest support in the report of one of the teachers who taught her above average first grade children the subtraction algorithm with and without regrouping. She reported that this teaching went much more

easily than it ever had before and that the children really understood regrouping this year. Her explanation for this was that usually the children struggled with subtraction combinations and these ordinarily get in the way of their learning the new algorithmic procedure. In this case the children used counting up to with finger patterns to do any subtraction combinations they did not already know. Thus they were freed to concentrate on learning and understanding regrouping and the subtraction algorithm.

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FIRST GRADERS' SOLUTION STRATEGIES OF ADDITION AND SUBTRACTION WORD PROBLEMS

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Summary

The present study is part of a research project on the development of children's ability to solve elementary arithmetic problems. In one longitudinal investigation empirical data were collected on the problem representations and solution strategies of thirty first graders who were given a series of simple addition and subtraction word problems (Verschaffel, 1984). In this paper some data on children's solution strategies and on the influence of the problem structure on those strategies are presented. The results are compared with the findings of other recent research, especially with those reported by Carpenter & Moser (1982).

INTRODUCTION

The current research on children's representations and solutions of simple addition and subtraction word problems is strongly influenced by the work o Greeno, Heller and Riley in the late seventies (Heller & Greeno, 1978). The se investigators introduced a new classification scheme for simple addition and subtraction word problems based on the semantic relations underlying th se problems. More specifically, they distinguish three main categories of verbal problems: change, combine and compare problems. Each of these three classes is further subdivided in distinct problem types depending on the identity of the unknown quantity and/or the direction of the event or relation. Greeno c.s. also developed a model of elementary arithmetic word problem solving, in which semantic processing is considered to be the crucial component in skilled problem solving. According to this model, the problem

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solving proces starts with the construction of an internal representation of the relations between the known and the unknown quantities in the problem in terms of one of the three different semantic schemata, namely change, combine of compare. Based on that representation, the problem solver then selects an appropriate arithmetic operation to find the unknown quantity. The model can execute but two operations: either a direct addition or a direct subtraction with the two given numbers. According to the model, the appropriate arithmetic operation is sometimes associated directly with the problem representation; in other cases the operation is selected after a transformation of the original problem representation to a combine structure.

In 1978 Carpenter & Moser started a longitudinal study on the development of the strategies used by children to solve different types of addition and subtraction word problems during the first years of formal schooling. In this study each child was individually interviewed eight times. Each time it was administered two addition and four subtraction problems. Carpenter & Moser's (1982) scheme for classifying the solution strategies consists of two dimensions. First they distinguish between additive and subtractive strategies. Secondly strategies are arranged according to their level of internalization: material strategies based on direct modeling with fingers or physical objects, verbal strategies based on the use of counting sequences and mental strategies based on recalled number facts. The results of the Carpenter & Moser (1982) study suggest a somewhat different picture of children's processes for solving addition and subtraction word problems than the one that had been proposed in the analysis of Greeno c.s. Indeed, their findings indicate that children dispose of a rich repertoire of solution strategies and that they make use of many of those to solve various problems. Their data further yield evidence that young children's solution strategies for subtraction problems are significantly influenced by the semantic structure underlying a problem. More specifically, it was found that children operating on a material or verbal level tend to solve each subtraction problem with the strategy that corresponds most appropriately to the semantic structure of the problem. Almost all children used a separating from or a counting down from strategy to solve the separating problems; for the joining problems nearly all children used an addin

on or a counting up from given strategy, and for the compare problems matching was the most frequently used procedure.

Since 1979 we are also working on a research project about the development of children's ability to solve elementary arithmetic problems. In one longitudinal investigation empirical data were collected on the problem representation and solutions of thirty first graders. They were interviewed individually three times during the school year: at the very beginning in September, in January and at the end in June. Each time they were administered eight word problems: four addition problems (a change 1, a change 6, a combine 1 and a compare 3 problem) and four subtraction problems (a change 2, a change 3, a combine 2 and a compare 1 problem)[a]. In the rest of this paper we will present briefly some results concerning children's solution strategies and the influence of the problem structure on these strategies. Generally these findings are not in conflict with the above-mentioned results reported by Carpenter & Moser (1982), but they rather complement them. A discussion of some other data relating to the problem representations of our subjects can be found in De Corte & Verschaffel (in press).

MATERIAL STRATEGIES FOR ADDITION PROBLEMS

As stated before, Carpenter & Moser (1982) found a significant influence of the semantic structure of subtraction problems on children's material strategies. For addition problems such a relationship was not found. Nevertheless these investigators admit that theoretically there are different ways in which the material additive strategy - counting all with models - can be carried out. They also observed clear examples of these types of strategy. But they conclude that young children do not distinguish them in solving different types of addition problems (Carpenter & Moser, 1982, p. 14).

In contrast to Carpenter & Moser (1982) we found a clear relationship between the semantic structure of addition problems and children's material solution strategies. Almost all children who solved the change 1 problem ("Pete

[a]The names of the problem types are borrowed from the classification scheme for addition and subtraction word problems of Riley et al. (1983).

had 3 apples. Ann gave Pete 5 more apples. How many apples does Pete have now ?") materially, applied the following strategy: they constructed a set of 3 blocks, then added a set of 5 blocks and finally counted the total number of blocks, giving the result as the answer. The combine 1 problem ("Pete has 3 apples. Ann has 7 apples. How many apples do Pete and Ann have altogether ?") was solved mostly in a slightly different way: the majority of the children constructed two distinct sets of 3 and 7 blocks respectively; they counted the total number of blocks either after moving together the two sets or without physically joining them. Most children solved the change 6 problem ("Pete had some apples. He gave 3 apples to Ann. Now Pete has 5 apples. How many apples did Pete have in the beginning ?") in one of the above-mentioned ways. But some applied another strategy: first they constructed a set of undefinite size; then they took away 3 blocks; next they increased or decreased the originally constructed set until it contained 5 blocks; finally they counted the two sets (3 and 5) and gave the result 8 as their answer. About half of the children who solved the compare 3 problem ("Pete has 3 apples. Ann has 6 more apples than Pete. How many apples does Ann have ?") at the material level, used the same strategy as for the change 1 or the combine 1 problem. The other half used another strategy: they constructed a set of 3 blocks, then built a second set of 3 and 6 blocks, and finally counted the total number of blocks in the second set, which yielded the answer 9.

The foregoing shows that the semantic structure of addition problems strongly influences the solution strategies of children who are operating on the material level.

VERBAL STRATEGIES FOR ADDITION PROBLEMS

In their classification scheme for additive strategies Carpenter & Moser (1982) distinguish between counting on from first and counting on from larger. In counting on from first the child begins counting forward with the first addend in the problem; the counting on from larger strategy is identical except that the child starts with the largest of the two given numbers. For problems in which the the largest addend is also the second one, counting

on from larger is the most efficient verbal counting strategy. In their reports Carpenter & Moser (1982) do not mention any effect of the semantic structure on the kind of verbal counting strategies children use to solve addition word problems.

Contrary to Carpenter & Moser (1982) we found that the frequency of the more efficient counting on from larger strategy varied with the semantic structure of the addition problems. For example, the combine 1 problem ("Pete has 3 apples. An has 7 apples. How many apples do Pete and Ann have altogether ?") elicited much more counting on from larger than counting on from smaller strategies; for the change 1 problem ("Pete had 3 apples. Ann gave Pete 5 more apples. How many apples does Pete have now ?") we found the opposite. We think that this finding can be explained as follows. To obtain a mapping between the internal semantic representation of an addition problem starting with the smallest of the two given numbers, and the counting [illegible] from larger strategy, the child must interchange the two given quantities in his problem representation. This reorganisation of the original problem representation is more radical when the two sets have a different function in the problem as in change problems (start set and change set) than when their function coincides as in combine problems (both are subsets) (see also Riley et al., 1983, p. 185).

MENTAL STRATEGIES FOR ADDITION AND SUBTRACTION PROBLEMS

Carpenter & Moser (1982) do not report empirical data showing that the semantic structure also influences the solution strategies used by children operating at the highest level of internalization, namely the mental level. This is not surprising, when we take into account that in their classification scheme hardly any distinction between mental strategies is made. We developed a more elaborated scheme for mental solution strategies. This enabled us to explore relationships between the problem structure and the solution strategies, not only on the material and verbal, but also on the mental level. We will illustrate this for subtraction problems.

In Carpenter & Moser's (1982) classification scheme different material subtraction strategies are identified: separating from, separating to and adding

on. They also distinguish parallel strategies based on verbal counting: counting down from, counting down to and counting up from given. In our classification scheme we extended this tripartite distinction to the mental level: (1) strategies in which the answer is found by subtracting the smallest given number from the largest one (direct subtractive strategies); (2) strategies in which the child determines what quantity should be subtracted from the largest given number to get the smallest one (indirect subtractive strategies); (3) strategies in which the child determines what quantity the smallest given number must be added with to obtain the largest one (indirect additive strategies). Our empirical data support this distinction and they yield evidence that also on the mental level the semantic structure influences children's solution strategies. The change 2 problem ("Pete had 12 apples. He gave 4 apples to Ann. How many apples does Pete have now ?") was solved mostly with a direct subtraction strategy (e.g. "12 minus 2 is 10 minus 2 is 8"). In contrast, the change 3 problem ("Pete had 5 apples. Ann gave him some more apples. Now Pete has 14 apples. How many apples did Ann give to him ?") was solved very rarely with an direct subtractive strategy. The great majority of the children operating on the mental level applied an indirect additive strategy (e.g. "5 plus 5 plus 4 equals 14, so the answer is 9").

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RESEARCHERS TAKING IN CHARGE A CLASS: RESULTS OF A THREE YEAR PROJECT ON NUMERATION

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We shall present a few items proposed to two groups of children in interviews and compare their understanding of selected aspects of numeration over a three year period. One group was given regular instruction on numeration, from grade one to three; the other group was taught by the researchers, from grade one to three, during the same period. The researchers main strategy was to bring the children to develop their own procedures and representations in view of constructing a meaningful and efficient number symbolism.

In a previous research study on numeration we had developped instruments, whic could be used as curriculum guides, to help construct learnings favoring the understanding of numeration and also used to constructed items to detect children' difficulties and to identify their acquisition of numeration. To be sure that these materials were effective for teaching we decided to undertake teaching a class and to apply them to a classroom in an elementary school.

From the winter of 1981 to June 1983 we took in charge the math curriculum wit a class of 25 children. We followed this same group of children over a three year period, from grade one to grade three (from age 6 to 10). During the winter of 1981, we worked with the two grade one classes in the school. The division of the children into two grade two classes was made by the principal and the teachers who were directly involved with the children in grade one. They made the split as even as possible. Our group of children was an "ordinary" group similar to groups in any school. From this time on, we work with only one of the two classes. We assured the principal, teachers and pare that by the end of the school year the objectives of the regular math program would have been covered. We also gave assurance that the children would be ab to pass the four school board exams given during the course of the school year

Math classes, lasting 50 minutes, were held four days a week. Nadine Bednarz was the teacher for two of these periods, and Bernadette Janvier the teacher for the remaining periods.

By this involvement as teachers in classrooms we developed teaching procedures which proved to be effective for teaching, and this school interaction helped us to refine and improve the tools and ideas developed in our former research on numeration started in 1976.

The research study undertaken in 1980 was conducted with 200 children, of age six to ten. The analysis of interviews given to all of these children permitted us to identify children's difficulties and errors, and in particular revealed to us important facts about their conception of numeration, constructed under the traditional instruction they had received (1) (2).

These observations were interpreted in the light of previous work on: 1) the mathematical analysis of the concept, 2) its evolution in the history of mathematics, and 3) the perception of numeration found in school textbooks.

All this led us to develop a theoretical framework on numeration, and to the production of a "reference framework" which we later used during the three years in our role as teacher/researchers (1).

The teaching, which we undertook with the same group of children over a period of three years (from 1981 to 1983), was in accordance with our conception that numeration is a concept that is constructed. Numeration is seen by us as a process of representation of number provoked by the need to communicate information on a collection and to operate on it. This process is characterised by a series of transformations on a collection accompanied, at the same time, by representations that are constantly reworked and modified. In this series of transformations the passage to the regrouping of a collection is an important step. The realization of this process is a representation of number that is able to be efficiently used for any collection for purposes of communication and operation, and ultimately this is the positional representation of number.

In the learning of numeration based on this conception the child should live situations which invoke this process, and that eventually leads to this ultim representation.

In our teaching strategy we confronte the child with the necessity for regrouping. From the start they work with two distinct groupings. Communication plays an important role and the operations are an integral part of the learning of numeration for, the evolution from a descriptive representati to an operational representation is highly related to the efficiency of calculation (3).

We create "contextual environments for learning" - a series of situations al bases on the same material. Its from the material that children develop thei representations. When presenting a situation to the children, we have to be careful that the material or the representation used in it are accessible. Three criteria help us determine the accessibility. First, groupings present are clearly identifiable. Second, the relation between presented groupings i explicite. Third, the actions to apply to groupings are direct, and do not require substitution (3).

In addition to teaching from 1981 to 1983 we also constructed a set of items, for each grade (using the "reference framework") which were used in intervies at the end of the year. These items were given to each of the children in ou group and also to a group of children from another school (that we shall call "Cohort Y"), in the same school board. Both groups had to meet the same obje tives and write the same four school board exams. The math teacher in Cohort was the regular teacher for that school year.

We had interview data from three sources (see Table 1 below).

[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	Cohort I (Traditional Teaching) Cohort I (Constructivist Learning) [illegible] [illegible] [illegible]

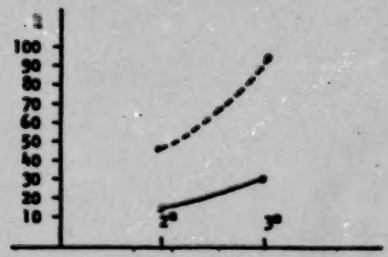
These interview data were used in comparative studies in view of answering a series of precise questions. In particular:

1) How do children in School Board A (1980) compare to the children, three years later, in Cohort Y on the following facts?
   a) see that conventionnal writing is related to collections "structured in groupings;
   b) make operations involving doing and un-doing groupings
   c) work with two groupings of different category at the same time
   d) use groupings as the "best" strategy to communicate information or operate on a collection
   e) make correspondances between two systems of representations with which the previously gained familiarity
   f) transfer from familiar systems of representations to unknown and less accessible ones.
2) How do children in the constructivist learning group compare to the children in the traditional approach on these same facts?
3) How did children in Cohort X and Y evolve in their understanding of numeration over the three year period?
4) How do children in the constructivist learning group compare to children in the traditional teaching group on attitudes, the strategies and procedures used, etc.?

We shall try to answer these questions in our presentation by selecting items which permit us to observe these facts. Due to limits of time we shall only discuss three of these facts.

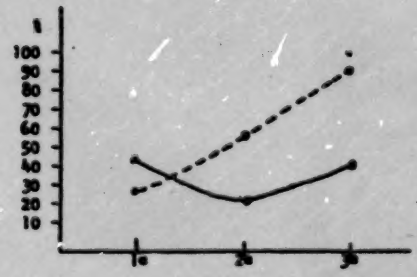
In our discussion we will refer to the following summary results.

Concerning facts d) and e): % of those using groupings as the "best" strate to communicate or operate and see the conventional writing as related to collections "structured in groupings":



It was interesting to see the different ways the children organized their wo once they had seen the pertinence to regroup.

Concerning fact b): % of those making operations involving un-doing groupin



We can see that at the end of the third year almost all of the children in t constructivity cohort saw the regroupings (in a context where these are rela vely accessable) and could operate with them.

Concerning fact f): transfert of squired habilities in situations where the rule of regrouping is given by a convention (this wasn't worked during the 3 years):

	Constructivist learning	Traditional teaching
Subtraction (random) 1□3 | 40 - 1□7 | 80		
Relation between groupings is hidden. Those un-doing the groupings.	92%	38%
Subtraction (shown à vigor) 7152 - 138		
Relation between groupings is given by a convention. Those doing the exchange on the sheets	58%	9%
Those calculating, taking away 1 ten and adding 2 units	9%	38%
Subtraction (conventional writing) 234 - 178		
Those giving a signification to borrowing in terms of groupings	94%	23%

We note that 50% of the children in our group can transfer the acquired abilities to a situation where the representation is less accessable, and make a corresponding effective action. Similar results were obtained for division.

Clearly from the above table, the children's learning is not completed, and if the teaching had continued we would have proposed "contextual environment for learning" where the relation between groupings is given by a convention.

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UNDERSTANDING MULTIPLICATION AND DIVISION

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Abstract: Three investigations with 12-13-year-olds of average ability are reported, looking at some aspects of understanding of multiplication and division of positive decimals in various contexts. In many details, the results are consistent with previous findings, and with Fischbein's theory based on the notion of intuitive primitive models for the operations.

The multiplicative relationship between three quantities occurs in a wide variety of contexts, including speed, price, measure conversion and enlargement. In all of these cases, multiplicand and multiplier can be distinguished. For example, in enlargement the scaling factor is naturally seen as a multiplier which transforms the original length. For speed and price (which we label "rate" structures) the multiplicands are speed and unit price and the multipliers time and quantity, respectively. In the simple case number per group X number of groups = total, two types of division are distinguishable, namely partition (division by number of groups) and quotition (division by number of objects per group). For the cases listed above, this distinction can be generalised, so that any division of the form product/multiplier is called partition, and any division of the form product/multiplicand is called quotition.

All the cases considered so far are classified as examples of "isomorphism of measures" by Vergnaud (1983). There are other applications of multiplication in which the quantities multiplied do not show asymmetry. These include Cartesian products, and area and volume (classified by Vergnaud as "product of measures") and defined quantities such as man-hours (classified as "multiple proportions"). Thus Vergnaud's classification is in terms of mathematical structures, whereas ours is a more informal and less comprehensive one based on some of the most frequently encountered contexts. Other applications of multiplication and division not considered here are those which involve composition of operators (e.g. one scaling factor followed by another).

Fischbein's work is along the same lines as ours and he has put forward a theory to account for the results obtained from this and related research. It is proposed that arithmetical operations are intuitively associated with primitive models which have behavioural origins. These impose restraints in relation to the combinations of numbers encountered. For multiplication, the primitive model is repeated addition. For division, there are two - partition and quotition - although there are some indications that the latter appears later. (For further details, see Fischbein (1983)).

CHOICE OF OPERATIONS

A paradigm which has been extensively used in the last five years is to present a verbal problem and to ask pupils to state which calculation would produce the answer, without computing it. One reason for interest in such questions is the availability of calculators. (Actually, this paradigm has been around for some time - see Greer, 1983). In a recent experiment, choice of operation tests systematically compared understanding of multiplication and division in four contexts (speed, price, measure conversion and enlargement). Groups of 78 and 66 12-13-year-olds of average ability were tested with the same combinations of numbers across contexts, and, within contexts, for both types of division. Space permits only a summary of the results.

Multiplication items

1. Levels of difficulty were approximately equal for the four contexts.
2. Within contexts, there was a consistent pattern whereby the type of number used as multiplier was the major determinant of difficulty whereas changing the type of the multiplicand had little effect. Three levels of difficulty could be distinguished, in terms of the type of number used as multiplier, namely integer (92% overall correct choice of operation), decimal greater than 1 (71%), decimal less than 1 (53%).

Division items

1. Overall, the percentage of correct choices was much lower than for multiplication (33% v 75%). Price problems proved markedly easier than those for the other three contexts (presumably reflecting familiarity).
2. No consistent difference between partition and quotition problems emerged,

but the pattern of errors showed an interaction. For partition questions overall, 34% of responses were reversals and 15% were multiplications; precisely the reverse was true for quotition questions.

Detailed results from both types of item were consistent with the now familiar numerical misconceptions. The multiplication results provide strong support for Fischbein's theory that multiplication is mediated by the primitive model of repeated addition. The patterns of errors for division are also in line with the theory. For two of the calculations used, 26 ÷ 3.25 and 12 ÷ 0.75, he predicts that reversals should be more likely for partition questions than for quotition; this is indeed what was found.

There are many problems with this methodology, of course. One is the choice of numbers (in future work we plan to switch from using the same numbers to varying numbers randomly within a class assumed to be equivalent). Another very important factor is the precise wording used - some specific features of the results can be tentatively identified as artefacts of this.

MAKING UP STORIES FOR CALCULATIONS

To produce another piece for the jigsaw, children were asked to write verbal problems (stories) to fit given calculations (Bell, Fischbein and Greer, 1984). A weakness of that study was that story questions were interleaved with choice-of-operation questions, so the latter may have influenced the former. In a follow-up study, 50 children (12-13-year-olds, average ability) were given 4 multiplications and 6 divisions to write stories for. The percentages of stories judged appropriate were: 18 X 0.50 (24%), 4 X 0.70 (30%), 6 X 1.25 (40%), 0.42 X 0.36 (4%); 12 ÷ 3 (24%), 12 ÷ 0.50 (8%), 3 ÷ 0.75 (8%), 4 ÷ 24 (8%), 7 ÷ 9.52 (0%), 13.88 ÷ 3.47 (4%). Major features of the results were:

1. For multiplication items including one integer, the integer was always used as the multiplier in correct responses.

2. As can be seen, the only division item handled with any degree of success was 12 ÷ 3 ; of the .. responses judged correct, 22 were partition divisions, and 5 quotition.

3. The most popular context was money, followed by weight and speed (c.f. Ekenstam and Greger (1983) who report money, followed by weight and length).
4. Numerous findings reported in Bell, Fischbein and Greer (1984) and in Ekenstam and Greger (1983) were replicated - for example: "forcing" calculations into inappropriate forms (e.g. using decimals for what logically could only be integral); reversals of division (especially for $4 \div 24$) to make the problem easier; "pseudo-stories" (e.g. "A person wanted to find the result of $15 \div 2.5$. Find the result for him" (Ekenstam and Greger, p.381)).

INTERVIEW STUDIES

Two types of exercise were used in videotaped interviews with 12-13-year-old children of average ability, in group of 2 or 3. One type was directed at numerical misconceptions, and the starting-point for discussion was a card with the following questions:

4	×	3	=	12
FIRST NUMBER		SECOND NUMBER		ANSWER

In this multiplication sum the answer is bigger than the first number and also bigger than the second number.
1. Can you find a multiplication sum where the answer is smaller than the first number but bigger than the second?
2. Can you find a multiplication sum where the answer is smaller than the first number and smaller than the second number too?

A second card was used later, posing similar questions for division. The aim was to proceed from examples generated by experiment (the children had calculators) to statements of the general rules. In general, the children required a great deal of pushing to make any progress and showed a marked reluctance to experiment. A frequent observation was that having found one example for a particular question, they tended to generate further examples closely related to it, and to formulate highly restricted hypotheses.

The second type of exercise was designed to explore non-conservation of operation i.e. lack of awareness that the relationship between quantities is invariant under changes in the number (c.f. Ekenstam and Greger). One of the starting-points read as follows:

> An explorer went to visit a tribe's village. He drove the first 25 miles of the journey along a rough road at a steady speed of 20 miles per hour. He had to go the next 3 miles by canoe, at a speed of 6 miles per hour. Then he had to cut his way through 12 miles of dense jungle and could only manage a speed of 0.75 miles per hour. Finally, he came to a track and walked the last 15 miles at a speed of 3 miles per hour. How long did it take him altogether to reach the village?

Even with this juxtaposition of the four separate calculations, the children were oblivious that the same operation applies throughout. For the first calculation, the usual solution process was by ratio arguments, or by splitting the 25 miles into 20 and 5. For the second, answers were instantaneous, based on ratio arguments. The third led to the familiar pattern whereby the children correctly inferred by qualitative reasoning that the answer would be more than 12, and then wanted to multiply since "multiplication makes bigger" (c.f. Ekenstam and Greger's example, p.376). The fourth was again instantaneously solved. Even with a lot of argument it was difficult to persuade them that all parts could be calculated in the same way (again, c.f. Ekenstam and Greger, p.377).

CONCLUSIONS

The ability to use the four basic operations with positive decimals and to understand a range of applications of the operations seems a reasonable objective for children of average ability. Research such as that reported here shows that this objective is not being achieved and identifies some specific

problems and possible causes. Teaching experiments at the Shell Centre in Nottingham have shown that misconceptions are deeply rooted and hard to eradicate, and that it is not easy to establish models of the operations rich enough to cope with the cases we have been looking at.

The results reported here are consistent with previous findings and strongly in accord with predictions from Fischbein's theory. However, the causal framework of that theory is debatable. Fischbein states that the intuitive models impose numerical restraints. Might it not be more accurate to reverse this i.e. to say that the children's models are shaped by the properties of the instances they experience? For a long time, the arithmetic they do follows certain rules. In particular, they begin by learning the operations for the natural numbers - this is probably unavoidable. The pedagogical problem is how to help them to move from the limited conceptions which result to more general and powerful ones.

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TASKS TO MEASURE THE DEVELOPMENT OF COMPUTATIONAL ESTIMATION SKILLS AND CONCEPTS

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The purpose of an NSF funded project beginning summer, 1984, is to trace the development of estimation skills and concepts by analyzing spontaneously developed estimation strategies used by children in grades three through eight. Eight children at each grade level will be presented with estimation tasks, and their responses videotaped for later analysis. This analysis will be a method suggested by Case in his work on a developmentally based approach to instruction. The basic estimation tasks are now being designed and are scheduled to be piloted in September. The tasks themselves and the piloting results will be the focal point of the paper presentation and ensuing discussion.

The importance of developing computational estimation skills has recently received a considerable amount of attention. The National Council of Supervisors of Mathematics (1977) included estimation and approximation as one of ten basic skill areas in need of development. The National Council of Teachers of Mathematics, through the PRISM report (1981) and the Agenda for Action (1980), along with the Conference Board of the Mathematical Sciences (NSB Commission, 1982), in its report on what is fundamental in the curriculum, recommended substantially more emphasis be given to estimation. In fact, the National Council of Teachers of Mathematics will emphasize estimation in its 1986 yearbook.

Such endorsement is long overdue. Skill in estimation is a useful pedagogical tool in developing problem solving, number sense, mental arithmetic, and alertness to the reasonableness of results. It is often more useful than precise calculation in everyday situations. Finally, as usage of calculators increases, so does the desirability that students recognize input errors through estimation of the calculations.

It is widely recognized, however, that the teaching of estimation skills is one of the most neglected areas of the mathematics curriculum (Buchanan, 1980; Trafton, 1978). As a result, both children and adults are frequently poor at estimating computational results (Underhill, 1983). Results from the second NAEP study underline this probable cause-effect relationship (Carpenter, Corbitt, Kepner, Lindquist & Reys, 1980).

To date, the bulk of the research on estimation has concentrated on relationships between estimation ability and other abilities (Bestgen, Reys, Rybolt & Wyatt, 1980; Hall, 1977; Levine, 1982; Paull, 1971), comparison of methods of teaching estimation (Bestgen et al., 1980; Hall, 1977; Schoen, Friesen, Jarrett & Urbatsch, 1981), and identification of estimation strategies (Reys, Rybolt, Bestgen & Wyatt, 1982; Threadgill-Sowder, in press). There has been almost no research directed towards the development of estimation skills or the causes underlying difficulties encountered by students while attempting to estimate even though Carpenter and his colleagues (1976) have asserted that estimation encompasses a variety of mathematical skills which require understanding of basic mathematical concepts.

The overall purpose of this project is to clarify the complex of skills and concepts which are related to and/or are a part of the ability to estimate, and ways in which these skills and concepts develop. The theoretical basis for the procedures to be used is the developmentally based theory and technology of instruction as developed by Case (1978). Case has identified three characteristics of tasks which would benefit from the application of this developmental approach. The characteristics are: (1) The task should be a difficult one for students to master, given current methods of instruction; (2) the task should have a specifiable objective and method of solution; and (3) the task should be one which elicits consistent responses from those who fail it (Case, 1978, p. 458). Estimation tasks seem particularly appropriate for Case's developmental approach.

Basically, Case's procedure involves analyzing children's spontaneously generated strategies, by presenting tasks to children, observing the way in which they solve the tasks, then generating and testing hypothesis concerning

the nature of the strategies. In the projected study, eight students of average computational ability at each of grades three through eight will be identified through scores on a computation test. Four of each group of eight will be individually presented with estimation tasks in a structured interview setting. Videotapes and transcripts of the interviews will form a data base from which to generate hypotheses about the development of children's estimation strategies and associated skills and concepts. Accordingly, task will be revised, the remaining students interviewed, and transcripts and tapes again analyzed. Finally, a developmental model of the growth of estimation skills and concepts will be formulated. It is expected that researchers interested in the development of mathematical ideas, and practitioners, in instructional materials for school use, will find the results of this project relevant to their work.

The first major phase of this project is the development of the initial tasks to be used in the interview settings. The design of the tasks is of the utmost importance. Careful consideration must be given to the working memory requirements of the tasks, to the familiarity of the items of information to which the child must attend, and to the salience of stimuli to which the students must attend (Case, 1978, pp. 442-446).

The presentation associated with this paper will demonstrate the initial tasks designed, and discuss piloting results. It is hoped that during the ensuing discussion critical feedback on the tasks will be offered by other researchers.

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THE USE OF OPERATOR FUNCTIONS TO TEACH RATIONAL NUMBER CONCEPTS

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The purpose of this project was to examine the use of an operator model to teach the concept of rational numbers. Several concrete embodiments were used for which inputs and outputs into machines represented rational numbers. Initially chips were used, but the model was expanded to include other concrete materials such as currencies (e.g. 3¢ for $1), bicycle gears, shadows and consumer problems. (e.g. 8 bottles of pop for $1.20). The concrete models were generalized to abstract notation such as products of rational numbers, equivalence of rational numbers and comparison of rational numbers.

The results indicated that the children did learn significantly more than those in a control group. The observation throughout the project and the interviews with the children led to information which might be of use to teachers. The children moved to abstract algorithm naturally, but if they needed to return to a concrete modal, they usually returned to the first one they were taught.

Teaching about rational numbers in the elementary schools creates many issues which need to be resolved. To simplify discussion for this paper, th issues will be reduced to two general dimensions:

I. The mathematical conceptions of rational numbers,
II. Instructional models used to teach the concept.

Kieren (1976) identified five sub-constructs which he argued are used as m ematical representations of rational numbers. (1) part-whole relationship (2) measure relationships, (3) ratios, (4) quotients and (5) operators. was Kieren's model which was used by Behr, Lesh and Post (1979) in their i vestigation of the use of concrete materials for learning rational number concepts. The principle investigators in the project developed a comprehe sive set of instructional materials which used concrete manipulatives for teaching number concepts, but they did little with the use of operator fun

Flener and Lockwood, acting as interested mathematics educators, but not as formal participants in the project, developed a set of instructional activities using operator functions.

OPERATOR FUNCTIONS

Rational numbers can be developed in various branches of mathematics, analysis, algebra or geometry. For example, rational numbers may be defined to be solutions to the algebraic equation of the form $ax = b$, where a and b are integers, or the concept may be defined geometrically using principles of Euclidean construction. The idea of an operator function is based on a function approach to the development of rational numbers. That is rational numbers are defined as equivalence classes of ordered pairs of integers.

As a concrete model to represent operator functions, Dienes (1971) and Braunfeld (1969) suggest the use of "machines." The students assume that objects such as sticks or chips are put into the machines and there is then a specified output from the machines. Hence, sets of inputs and outputs for any machine can be formally considered equivalence classes of ordered pairs of integers.

ACTIVITIES

For this project the children initially used machines similar to ones shown in Figure 1.

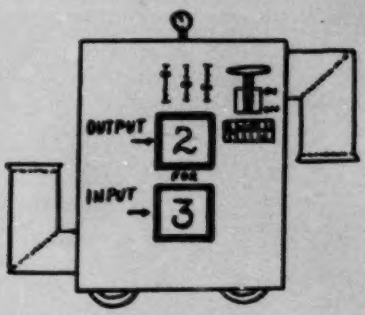


FIGURE 1

An Example of a Machine Used in the Project

For example, they were told that in a "2 for 3" machine for every three chips put into the machine, the machine would send two chips out. The use of the machine was extended to consider various amounts of possible puts. If 21 chips were put into the machine the children would have to determine that there would be 14 chips put out by the machine. For purposes of this project certain numbers of chips could not be put into a chine. For example, in a "2 for 3" machine, numbers which were not multiples of 3, such as, 4, 5, 8 etc. were considered "unacceptable."

The model was not limited to only chips. The students used the concept a machine for exchanging currencies from various countries. For example one bank might offer 5 German marks for 2 dollars, or 3 English pounds 7 dollars. Another bank can have different rates of exchange such as 7 marks for 3 dollars. Another model used bicycles as operators. The inp was the number of pumps on the pedals and the output was the number of t of the back wheel. Other models were shadows of colored rods created by overhead projector, and consumer type problems such as 8 bottles of pop $1.20.

Operators were also compared. A "5 for 6" machine was determined to be equivalent to a "10 for 12" machine. A bank giving 8 French francs for dollar was giving a better rate of exchange (thus, having a larger ratio determining a larger rational number) than a bank giving 15 French franc for 2 dollars.

The students were also introduced to graphical representations of the op tor functions. As an example, Figure 2 shows the graphs of the ten poss ratios for a bicycle having two gears on the sprocket and five gears on rear axle.

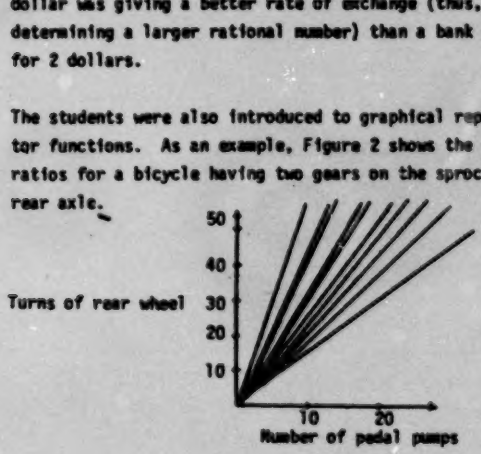


FIGURE 2

Graph of Gear Ratios of a Ten-Speed Bicycle

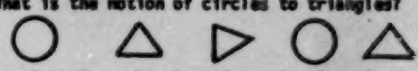
The rational fractions listed are determined by the number of teeth on the two gears which are engaged. The students were able to compare gear ratios using arithmetic by physically pumping on the pedals, and by visually examining the graphs of the gear configurations.

EVALUATION METHODS

The primary purpose of the study was to develop instructional materials and to form hypotheses, not to test hypotheses. However, one aspect of the full project was to develop evaluation instruments which could be used to measure students' understanding of rational number concepts in a variety of modes. It was decided that the instrument which was developed could be used to evaluate the students who participated with the use of the operator functions.

Three sample items are (easy, medium, difficult).

1. What is the ration of circles to triangles?



A. 3 to 2 B. 3 to 5 C. 2 to 3 D. 2 to 7

2. For two servings of hot chocolate you need 2 cups of milk; 6 teaspoons of chocolate mix.

   How many teaspoons of chocolate mix are needed to make 1 serving? _____

3. The fastest car in the world travels 8 miles in 2 minutes. How many miles does it travel in 5 minutes?

(Note: 4 of the 6 students in the experimental group answer 3 correctly, and only 1 in the control group answers it correctly.)

The test was developed in consort with Behr and Post's work with the other mathematical models. It was therefore somewhat surprising that the students who were using the operator models were able to produce reasonably high scores. Anecdotes were also recorded to give some indication of how the students were processing the material being taught, and following the written evaluation the individual students in the project were interviewed to determine how they

were solving the problems presented to them.

Six students participated in the experimental phase of the project. Six ditional students from the same class were matched on the basis of compa scores on a standardized mathematics test, and these students were used control group. At the time that the students in the experimental group were being taught, the other children were being taught fractions using their lar classroom textbooks. The two groups were compared on their performa on the evaluation instrument developed for the project.

RESULTS

Table 1 describes the difference in performance between the students in t control and experimental groups.

Table 1

Comparison of Control and Experiment Group using the Rational Number Test

Control Group	$n = 6$	$\bar{x} = 23.8$	$s = 3.9$	$^*t = 4.6$
Experimental Group	$n = 6$	$\bar{x} = 33.2$	$s = 3.2$	

$^*p < .005$

In examining the processes used by the students in doing problems it was termined that all of the children in the experiment used essentially the cognitive procedures. For example, if the children were to determine the output if 24 chips were put into a "3 for 8" machine they would use the f lowing general procedures:

(1) There are 3 chips out for every 8 which are put into the machin
(2) 8 chips are put into the machine--3 times. (This involves dividing 24 by 8 to have 3 groups.)
(3) Therefore there are 3 groups of chips, or 9 chips, put out by the machine.

Note that the "natural" solution to this problem, 3/8(24), is to divide f then to multiply. This is contrary to texts which first illustrate probl by multiplying numerators and denominators.

As the problems were presented in abstract format the student continued to use the same procedure as an algorithm. A solution to the problem (2/3)(18) would consist of dividing 18 by 3 then multiplying the quotient by 2. This was true for all six students in the experimental group.

At the completion of the project, the students in the experimental group were interviewed to help determine some of the cognitive processes they used when doing rational number problems. All six children, when questioned as to what concrete model they use if they need to think of a model, reported they thought of "chips." This model was the least meaningful of the ones used in the project. They actually exchanged foreign currencies, pumped the bicycle pedals, measured the shadows, and were very familiar with bottles of pop. It appears that the first model introduced has a powerful impact on the students and possibly this phenomenon needs further examination.

SUMMARY

The students in the experimental group were able to use the instructional material developed in this project and were able to transfer this knowledge to abstract problems involving rational numbers, and to also use their understanding to answer questions on a rational number test which was unlike anything they had been studying.

It was also learned that students can develop their own algorithm for doing arithmetic exercises and these algorithms may not be the same as those presented in a textbook.

Finally, there seems to be some extra significance given to the first activities learned by the students. They seem to prefer this embodiment to even more familiar ones.

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MENTAL COMPUTATION PROFICIENCY, ERRORS AND STRATEGIES OF 7TH AND 8TH GRADE STUDENTS ON TWO OPERATIONS WITH DECIMAL FRACTIONS[1]

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ABSTRACT

Two decimal fraction tests, a mental computation test (Math I) and a paper and pencil test (Math II) on multiplication and division operations were administered to 110 7th and 8th grade students. Mean performance of 13.58 and 16.65 were obtained on the two tests, respectively. The variability in individual scores were about the same. Standard deviations of 5.97 and 5.41 were obtained for Math I and Math II, respectively. A high and positive correlation coefficient (r) of .78 was computed, indicating existence of relationship between mental computation proficiency and paper and pencil performance. A number of errors were committed including dividing a bigger number by a smaller number "always". Prior work with whole numbers and common fractions seemed to interfere with students' performance on decimal fractions. Strategies used by these students to solve mental computation items were not different from those used to work out paper and pencil items. Multiplication items were "seen" as vertically presented and division items as one number inside the "box" and the other number "outside".

INTRODUCTION

Davis and McKnight (1980) and Lankford (1972) hinted at a possible relationship between proficiency at mental computation and paper and pencil mathematics tasks. Taylor (1980) suggested that the ability to do mental computation rapidly was an important skill in setting up and exploring possible solutions to a problem.

THE STUDY

The study being reported in here was an attempt to identify the errors, and strategies used by 7th and 8th grade students on the operations of

[1]The author would like to express his thanks to Dr. J. Fred Weaver for his comments on the earlier draft.

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division and multiplication with decimal fractions. It also sought to find out whether a relationship exists between mental computation proficiency (defined as the students' performance on Math I, a mental computation test), and performance on similar paper and pencil tasks.

METHOD

Subjects

A total of 134 7th and 8th grade students in two Wisconsin schools participated in the study. There were 9 7th and 125 8th grade students. A set of three tests were administered to this group of students in April and May 1984.

Tests

Three tests were administered in this study. A screening test, and two decimal fraction tests (Math I and II). The test questions were computational in nature. They were presented in horizontal format to avoid giving students a clue as to the method of attack.

Screen Test

This test contained 12 questions. To proceed to the next part of the study, students were required to get at least eight questions correct including two division items and two decimal fraction concept items. Of the initial 134 students, 18 were dropped from further participation in the study on the basis of their performance.

Math I (Mental Computation Test)

Math I was a mental computation decimal fraction test. It contained 24 test questions; 12 on division and 12 on multiplication. The students were required to work out the items "in the head" without writing anything down except the answer.

Math II

Math II was also a decimal fraction test. It contained 24 test questions similar to those on Math I. The only major difference between the two tests was that on Math II, students were allowed to write anything down if they so desired. Five students of the total 116 were absent on the

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day Math II was administered. Another student moved to another school district. This left a total of 110 students.

Procedure

The screening test was administered to all 138 students. The test was administered during mathematics class periods. Students satisfying the already indicated criteria were included in the main study.

The main study included the administration of Math I and Math II. The two tests in both schools were administered a week apart with Math I being administered first. The students' work was then corrected and rank ordered.

Interviews

Interviews with 25 "high" and 25 "low" students (those obtaining high and low scores on both Math I and II, respectively) were audiotaped for later analysis. Interviews were open ended, and on an individual basis.

RESULTS

Means and Standard Deviations

The means and standard deviations of the 110 students on Math I and Math II are given in Table I. Students' mean performance on Math II (paper and pencil) was higher than that on Math I. This was expected. Writing down steps on paper helped students keep track of steps to come next in the solution process. Although the means on Math I and II are different, the variability is about the same. Standard deviations of 5.97 (Math I) and 5.41 (Math II) were found.

TABLE I

MEANS AND STANDARD DEVIATIONS (MATH I & II)

Test	Mean	S.D.
Math I	13.58	5.97
Math II	16.65	5.41

Total number of students - 110

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Correlation Coefficient

A high and positive correlation coefficient (r) of .74 was computed, indicating that a relationship exists between mental computation proficiency and paper and pencil performance.

Errors

A variety of errors were committed by these 110 7th and 8th grade students. Generally, more errors were committed on the division rather than the multiplication operation. This finding is similar to that by Guiler (1946a,b).

One of the errors found involved dividing a large number (in value) by the smaller one regardless of their position in the question. Working out the item; .4 ÷ .8 = ?, N.E. a "low" student divided .4 into .8, giving the answer .2, keeping the decimal place of the question. On being questioned why she divided .4 into .8, she indicated that, ".4 was smaller,.....and the smaller always goes into the larger."

Another error made by these students was writing the actual remainder after the initial division process had been completed as the decimal part. Apparently students failed to realize the fact that it was possible to "complete" the division process with decimals by reducing further the "remainder" to tenths, hundredths, etc.

Another interference to students' understanding of decimal fractions seems to come from their earlier study of common fractions. This finding is similar to that found by Hiebert and Wearne (1983). Students wrote answers to some division items as .1 3/5 or 2.27/[illegible] clear intrusion of the common fractions.

Strategies

Almost all students (low and high) used the same strategies to solve both mental computation and paper and pencil items. Most of them pictured the multiplication mental computation items "in the head" as arranged in vertical format. The number with more digits above that with few digits. N.E. solving the item; .4 x 3.44 = ? went as follows: ".4 x 3.44 that's, mmm, 4 times 4 is 16; carry the 1. 4 times 4 is 16; plus 1, 17; carry 1; 4 times 3

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is 12. Plus 1 is 13. So it is 1376." Writes 1.376. On being asked how he pictured the item, said "just as I would write them on the piece of paper. This number over the other...." R.E. was from the "high" group. For division items, students "saw" one of the numbers in the "box" and the other "outside".

DICUSSION

Although a high correlation between two variables does not necessarily imply causation, it was generally found in this study that students who scored highly on the mental computation test scored highly on the paper and pencil test. A similar relationship held true for those who obtained low scores on the mental test.

Students generally did better on the written test than on the mental computation test, this was most obvious with the "high" students. It seemed as if the use of paper and pencil gave the "high" students more chance to organize their information for solving the question. For the "low" students, the use of paper and pencil did not seem to help much.

It was also found that the students, as a whole, gave many and different errors to the items in this study. Many of these seemed to be random and did not bear any resemblance to the correct answer. For these students, it seems (to me) as if even if calculators are provided to them when solving decimal fraction problems, they would not be able to ascertain whether the answer(s) they have obtained is/are reasonable. It seemed as if most of these students lacked complete understanding of the decimal fraction concepts.

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THE LEARNING OF ALGEBRA

FROM AN ARITHMETICAL TO AN ALGEBRAIC THOUGHT
(A clinical study with 12-13 year olds)

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Abstract.- Theoretical and historical considerations seem to indicate that there is a didactical cut in the evolution line that goes from an arithmetical to an algebraic thought [2]. There are several aspects of the individual way of thinking which suffer essential changes when passing through this cut. Some of these aspects have been pointed out by C. Kieran [4]. We have carried out a clinical study with a stratified sample of 12-13 year olds that reveals some other aspects and previous difficulties to the acquisition of the algebraic language, for instance, that the children's tendency to stay at an arithmetical level when solving the first "non-arithmetical equations" (linear equations with two occurrences of the unknown) makes manifest certain behaviors related with their reading and interpretation of literal notation, algebraic symbols and algebraic expressions and with their solving strategies choice which would have necessarily to be taken into consideration in any teaching approach of elementary algebra.

INTRODUCTION.- The elementary algebraic language has to be constructed in child on arithmetical basis; this construction will be possible from extending the elementary arithmetical operations to new other objects (for example, to the unknown and whole algebraic expressions) but, at the same time, breaking with previous arithmetical notions (for example, with the notion of equality). Both, the generalized arithmetics and the new algebraic conceptions begins to be developed in child, by means of teaching, in a moment that we define as a didactical cut, [see Filloy/Rojano, 1983] . It is the moment when child faces for the first time linear equations with occurrences of the unknown in both sides of the sign " = ". The purpose of the present study is to make clinical observations in children, in the moment of the cut, analyzing their spontaneous responses to the new kind of equations and correlated problems, and trying to detect those permanence behaviors in arithmetics which could be real obstacles for the algebraic language construction. The clinical study and its results are described below.

PREPARATION OF THE STUDY

The clinical study covered two generations of High School students of one secondary school throughout the years 82-83 and 83-84, respectively. Both interviewing periods were preceded by written diagnoses for measuring pre-algebraic efficiency. On the basis of the results therefrom obtained, the student population was stratified and a stratified sample was chosen for the clinical interview.

The theoretical considerations exposed in [2] led us to use the diagnostic test to determine the population's background on the three pre-algebraic subtopics, namely: i) literal notation equa-

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tions,that is linear equations of the form $X \pm A = B$, $A \times (X \pm B) = C$, $AX = B$ (A, B and C being natural numbers); and ii) non literal notation equations, for example, $\square \pm A = B$, $A \times \square = B$ (A and B, natural numbers); and iii) Semantics, that is, translation into equations, interpretation and/or solution of statements containing problems corresponding to equations i) and ii) and to "first non-arithmetical equations", that is those in which the unknown ($Ax + B = Cx$; A, B and C being natural numbers) already occurs twice. A specific stratification of the 82-83 generation was carried out for each subtopic on the basis of quantitative and qualitative criteria (according to type and number of items solved), which resulted in five strata for each subtopic. This time, the sample was constituted by children who had been consistently placed in one of the strata according to the three subtopics (see figure 1); only the extreme strata (1 and 5) and the intermediate stratum (3) were considered (figure 1, shaded areas).

Fig. 1

Thus, 6 children for the upper stratum, 4 from the intermediate one and 4 from the lower one were interviewed.

In the 83-84 test, the same subtopics were considered: i), ii) and iii). This time, a linear stratification for each subtopic or axis was also carried out —four strata fitting into each axis— and a spatial stratification was subsequently carried out in which each child was assigned an ordered triplet the coordinates of which correspond to the efficiency level in each axis, and is shown in a cube as the one in figure 2.

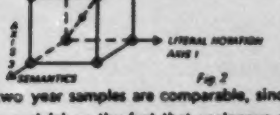


Fig. 2

The sample was made up considering mainly the strata or classes along the principal diagonal and the vertices of the cube.

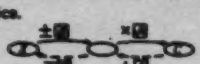
The two year samples are comparable, since the classes distributed along the principal diagonal correspond (given the fact that analogous criteria were applied) to strata shown in sample 82-83. The main difference roots off from the fact that, in 83-84, cases countermining the order of one or more axis were taken into consideration, for example, triplets (0, 0, 3), (3, 3, 0), (0, 3, 3), etc. Thus, both diagnoses cast a strata based view of the situation of the population surveyed, just at the moment which we have called didactical cut in the transition process from arithmetics to algebra, during which the interviews were carried out.

SCHOOL TREATMENT PRECEDING THE STUDY

As we mentioned before, we have worked with the population of a secondary school in Mexico City; here, the material used in teaching mathematics has been made up by a group of teachers from the Sección Matemática Educativa [1] . As regards second grade Arithmetics and

Algebra, this material intends to teach the student, first of all, solutions to the simplest equations ($X \pm A = B$ y $Ax = B$), as well as to problems arising therefrom, through the use of a scheme representing direct operation and inverse operation sequences for the solution.

From there we go on to equations and problems requiring two or more operations: $Ax \pm B = C$, $A \times (X \pm B) = C$, using the same device.



Once this has been solved, we go on to problems and equations of the type $X \pm A = BX$, that can only be solved by operating the unknown. We consider this the beginning of Algebra properly speaking, since it differs from Arithmetics in at least two aspects: on the one hand, the solving method requiring only arithmetical operativity with numbers —mainly concerning the relationship between elementary operations and their inverses— now transfers such operativity to the unknown (which up to now is only the representation of an unknown number); on the other hand, the equal sign relating arithmetical operations (left side) to their result (right side) acquires now a "more algebraic" connotation, since the unknown occurs on both sides and therefore the expressions which the equal sign links are more general and "equivalent" [4.] . Traditionally, this transition is not taken into consideration in teaching and, thus, the arithmetical method is extended to more general cases, disregarding the difficulties that the students may face in making such transference. The present study is to put forward such considerations.

The material for a clinical interview was prepared in order to observe the children's spontaneous attitudes when facing for the first time this new type of equations where the unknown must be operated, or which require a more evolved notion of equality than the arithmetical one, for example to procede to cancellation strategies.

CLINICAL INTERVIEW

It includes five different series of items: 1) E quation series, for diagnosis verification; this series includes equations of the form $X \pm A = B$, $A = B$, $A \times (X \pm B) = C$, and $(X \pm A) \times B = C$. 2) C cancellation series, to determine the level of "conceptualization" of the equation as an equivalence. It contains items such as: $X \pm A = B \pm A$, $AX \pm B = X \pm B$, $X \pm \frac{X}{A} = B \pm \frac{X}{A}$ and $X + X = A + X$. 3) Operating the unknown I series, in which we find items such as $AX \pm B = CX$ and $AX \pm B = CX \pm D$, to sound into students' difficulties when facing this new type of equations. This series largely includes an instructional section given by the interviewer, but referring to a concrete context. 4) Number finding A series, to analyze the solving procedures in the different strata. This series includes two kinds of problems: those which may be solved through equations of the form $X \pm A = B$, $AX = B \div A$ x $(X \pm B) = C$ and those which can be solved with equations of the form $AX \pm B = CX$. At this point, it is interesting to observe if the student

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can write the corresponding equation whether solving it or not. The problems are stated as a riddle, for example: "If we add five to a number and we subtract 13 and this equals 45, what's the number?" 5) Problem inventing P series, in which the student are supplied with equations of the forms described in A, and are requested to figure out statements of problems that can be solved by applying them.

This interview was applied just when the above-mentioned transition was taking place, that is to say, after the solving procedures to "arithmetical" equations ($X \pm A=B$, $AX=B$, $A \times (X \pm B)= C$) and corresponding problems had been taught, but before solving procedures for $AX \pm B = CX$ and $AX \pm B = CX \pm D$ had been taught. Thus, this was the first time that the students had been presented items as in C and A and in part of series A and P.

Twenty interviews from fifty minutes to two hours were carried out altogether. They were recorded in audio-video and reported in written by an observer. Also, all the sheets of paper used by the interviewee and the interviewer were collected thereafter.

RESULTS

The results of the interview were, in a general way, the following: Different behaviors for each stratum were recorded in each series of the interview. The most relevant were:

- In the diagnosis verification E series, the upper stratum (5) dealt successfully with the whole series by undoing the operations (inverting operations adequately so to find the unknown value). Most of the middle stratum (3), solved the one-step equation by means of a scheme described in a previous section. For the two-step equations ($4 \times (X + 11) = 52$) they used a mixed strategy: undoing the operation on the first step and applying "plugging-in"(non-systematic trial and error strategy) or specific fact procedures on the second one ($52 \div 4 = 13$ and "find a number which, added to 11, equals 13").

The procedures varied from solving by means of a scheme (not always successfully) to"plugging-in" (simple one as in $3X = 39$ or sophisticated one as in $6X = 37\,434$, where even magnitude order was used in "plugging-in"). In few cases the interview replaced children within the stratification given by the diagnostic test. The stratum strategies revealed that the processes tend to become automatic towards the upper stratum, whereas in the middle stratum there is a mixture of algorithm and "plugging-in", in the lower one, there is great tendency to approach each case as a new problem, instead of algorithmicizing the procedure.

- In the I series (first non-arithmetical equations), the upper and middle stratum children recognized them almost immediately as "new" cases, whereas the lower stratum children approached such items through "plugging-in" strategies, even in complex cases. Middle and upper stratum children set aside "plugging-in" throughout the first items of the series; then, the interviewer put forth an instructional section to solve such equations, whether based on a geometrical model or a

balance model. Some of the upper stratum children transferred spontaneously the steps follow when dealing with the models unknown to operating the unknown in the equation. This tr ference is not observed in middle stratum children since they cannot go beyond the model o in the complex cases. Regressive behaviors regarding solving procedures for equations of th series were encountered in middle and upper stratum children who had already mastered th but would "forget" them when such equations appeared as the intermediate stage of a more co plex procedure . . This may be due to the fact that, since children do not keep a written rec of the steps in the procedure, such simple equations do not appear to them explicitely, or a although the equation appears in written (for example, $4X = 16$), the concrete interpretation its elements in terms of the model used to solve it, is predominant. This model "centration" also encountered in cases where simple equations were proposed, out of context, after instruction was given.

a In the cancellation series, various ways of interpretating the equality are found: a) as an ar metical equality (I_{A_1}). Here, the student would try to "put together" or "read" the right-h terms with one only number before giving any answer or doing any operation; those who had t interpretation adopted one of the following procedures: i) by completion: "This (indepe ent term of the left side) lacks so much in order to be the same as this (complete right side ii) by "plugging-in"; interpretation I_{A_2} predominated in stratum (3), but was also present in two other strata. Items such as 2, 3, 4: $x + \frac{141}{10} = 17 + \frac{141}{10}$, $x + \sqrt{17} = 42 + \sqrt{17}$ and $x + \frac{1}{2} = 6 + \frac{1}{2}$ were particularly difficult to student having t interpretation; several students in stratum (1) not knowing the $\sqrt{17}$ value could not go on Item 3. b) The left side (as a whole) equals the right side (also as a whole);(C_3). This interp tation also admits two procedures: by completion (which is more visual than arithmetical) "plugging-in". In some of the middle stratum children, this interpretation (C_3) was followed interpretation I_{A_2} ; in other cases, in strata (1) and (5), it appeared alone. c) The equality ment to element (C_1). This interpretation allowed the students to solve fairly quickly the eq tions in this series, except items 4 and 12 which we will discuss later on. This interpretation dominated in stratum (5), even though in some cases it was preceded by interpretation C_3 or reshuffling of the elements in relation to the equal sign.

Items 4 and 12 in the C series: Item 4: $x + \frac{1}{2} = 6 + \frac{1}{2}$; Item 12: $x + 5 = x + x$ were given an terpretation of the C_1 type by almost all of the interviewed children; in Item 4, however, children assigned different values to the x having coefficient 1 and to the x in the numerator o The typical answer to this item was: "This x ($x + \frac{1}{2} = 6 + \frac{1}{2}$) is 6, and these ones ($x + \frac{1}{2} = 6 +$ can be any number". This answer occurred without variations in strata (5) and (3); it was found in stratum (1) with the assigning of a particular value (usually 1) to the x in $\frac{1}{2}$ as a va tion. In Item 12: $x + 5 = x + x$ the same situation as in Item 4 occurred: "This x ($x + 5 = x +$ is 5 and these ($x + 5 = x + x$) can have any value".

In some cases the children were asked to invent a problem which could be solved by means this equation; or else they were presented a problem having the same structure as in series A, make them understand that, in the context of a problem, all the occurrences of the unknown present the same thing; this explanation was not successful in all cases.

In the part of the A series which corresponded to an "arithmetical" equation, the follow

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procedures occurred: a) translation and solution of an equation. b) translation and solution of a scheme. c) solution of the problem without writing down neither the equation nor the scheme, inverting the operations directly from reading the text.

Procedures a), b) and c) in the three strata, without any of them being predominant. In stratum (1), it was almost always necessary to suggest the use of an equation or a scheme.

As to the part of this series corresponding to equations in which the unknown appeared on both sides of the equality, almost all of the interviewed children from the three strata were able to write down the corresponding equation, although some children in stratum (1) always tried to "guess" the answer before writing down the equation. In some cases, the series were used as a support to the C series, and were placed next to item 4 ($x + \frac{5}{6} = 6 + \frac{5}{6}$). In other cases it was used to verify if learning from solving equations with unknown on both sides (in series I) could be transfered to other contexts. This was only done with children in stratum (5).

Our last remark is that children placed on the cube's vertices, in the classification of the 83-84 test's results, did not show different behaviors from the ones already described, except for children of coordinates (0, 3, 3) or (0, 3, 0), whose poor performance in dealing with literal notation equations was due to the fact that they could not decipher correctly the symbol Ax (A being a constant); in one case they assigned x a digital from 0 to 9 to "complete a two-figure number" (4X = 40, so X =0), or else they assigned an arbitrary operation to this symbol (addition or multiplication between constant and unknown).

CONCLUSIONS.- The permanence in an arithmetical reasoning level appeared more pronounced in low stratum children than in others; this could be due to the fact that children who don't master algorithmic technics in solving the simplest equations (items for sequence E) and use an intuitive approach in these cases, do not realize that they are facing a new kind of equations; in a natural way, these children approach to new situations with the same naive primitive arithmetical strategies, as allow them to exhibit unexpected abilities in solving complex items. Whereas, middle and high strata children, who do see the difference between routine and new equations suspect that there should exist systematic solving procedures for the new ones, as allows them to stop using intuitive methods. Nevertheless, the regressive behaviors of loosing skillness, detected in middle and high strata children, may be seen as a return to a primitive arithmetical level. Both, the pramanence in or the return to this level could become real obstacles in the process of constructing the new algebraic language.

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CONSTRUCTING THE NOTION OF VARIABLE USING HISTORY AND BOTTLES

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The concept of variable is analysed using an historico-epistemological approach. It is suggested that the concept of variable cannot be developed without a graphical means for apprehending variation. Two complementary "séquence didactique" were designed and administered (interview) each to five groups of two children aged 13-14. They involve sketching filling curves of beakers and bottles, making each time a prediction. The testing sequence was aimed at comparing how each group could sketch on a micro-computer drawing pad (or tablet) motions appearing on the monitor. Main strategies and basic obstacles are described.

A conference organised by SLO (Foundation for Curriculum Development) at Enschede in February 1982 has shown that although the notion of variable is most important in mathematics programme not much research has been carried out in this domain.

The notion of variable is basically associated with the concept of function. However, we insist at the outset on making essential distinctions which focus on the evolution of the term over the past thirty years or so. Not very long ago, it was common to come across the expression "dependant variable" as the name of letters such as y, z, s which was assumed to "vary" in relation with an independant variable in general x. In fact, this independant variable was simply the image f(x) of an element x of the domain. To make things more precise, the term has denoted afterwards the rule through which y was assigned to x and not the result y. Actually, the formalised version of this idea of rule is the subset of A X B which we usually find in present-day textbooks. An "excess" or rigor (for most purposes) has led Bourbaki to formally characterised this "rule" by the triplet (A, B, C) where C is an appropriate subset of A X B, A and B being respectively domain and co-domain. This paper is mainly concerned with this idea of "rule". In JANVIER (1983) we mentioned that together with FREUDENTHAL (1982) we consider that behind the idea of "rule" stand several different types of objects. We refered to those objects as belonging to different semantic domains

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such as: transformation, permutation, sequence, isomorphism, variable. The epistemologico-pedagogical analysis which we undertake in this paper is only concerned with the semantic domain "variable" in other words with the "rule" in that it stands for "the way it varies".

HISTORY

Things change with time or otherwise. How can we describe those changes? How do we apprehend them? Nowadays, one can use cartesian graphs to suggest types of variation. But, how was it done before those graphs were invented? How can we apprehend temporal change without them? History can be used to examine this question. The following has been kept succint because of the limits imposed. At this point in time, I put forward the existence of two major landmarks in the historical development of the semantic domain of variable. The pre-graphical era ended roughly with ORESME (circa 1350). For more than 250 years, variation was analysed through diagram accompanied with full written description. This graphical era ended with Descartes who combined algebraic description (algebraically processable entities) to the diagrams. At that time, the power of the algebraic description of change and the study of fluxion by BARLOW and NEWTON paved the way to a basically algebraic approach to the study of variation.

From the pre-graphical era, two main facts seem to stand out. Clearly, descriptions of change were made through long and elaborate written (or spoken) explanations. The description of changes or of the relations between varying things were made rather indirectly through the use of proportions. Four elements were considered two by two. $\frac{V_1}{V_2} = k \frac{t_1}{t_2}$ was exclusively used and not $v = kt$ because ratio could not be assimilated to a number (the "V2-is-not-a-ratio drama"). This approach to variation was very indirect since it did not suggest one element varying with respect to another. It rather suggests a way to relate two ratios. This conception was certainly an epistemological obstacle (in the sense of BACHELARD, see BROUSSEAU (1982)). Indeed, the power of study derived from the use of proportion was an efficient knowledge, powerful but which prevents further development. We think however that this epistemological obstacle can be attributed mostly to a cultural and traditional features of greek philosophy. This

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difficulty would not be met with children developing the concept today.

As a corollary, this cognitive approach, it is said in GRIZE (1968) (and I think it has to be verified), led to a capacity to apprehend only linear relations. In fact, this feature is more remarkable since it corresponds with children to a precise obstacle which has to be overcome.

As for the second fact, it also characterised at least part of the graphical era. It is the intrinsic incapacity to see "number" or "quantity" as varying. In fact, even with ORESME great care is taken to make precise that his diagram show how quality varies. Actually, quantities (as I see it) were basically discrete and could not involve continuous change. This is another important feature for understanding pupil's construction of the notion of variable. Since discretising responses turn out to be numerous.

Let us mention a final fact before going to the experiment. During the graphical era, verbal descriptions kept being important in the apprehension of change. Indeed, the success come from a combined use of diagram and expressions such as: uniformly difform, difformly difform...

BOTTLE

This sort of analysis (too briefly presented here) leads us to hypothesize the development of the concept as the integration of a numerical "framework" into the notion to be examined. In other words, we envisage the development of the notion as an integration of two complementary cognitive processes: apprehending variation or motion with numbers or quantity and adding some dynamism into discrete set of numbers. Clearly, the integration is achieved through verbal activities. We see the role of language as being predominant.

On the basis of this analysis we thought of conducting two complementary "séquences didactiques" followed by a testing sequence. The first one (qualitative schema) consisted in asking pupils to plot directly on a transparency the filling curve of a becher. The transparency was taped on a plexiglass plate on which

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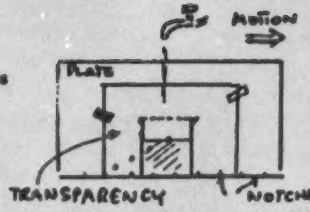
was cut on the lower edge regular notches allowing equal displacement every two seconds (see Figure 1). The becher stood behind and readings were taken straight in line with the water level.



The measuring scheme (quantitative scheme) consisted of plotting the same filling curve but using a measuring cup. The plotting was achieved through successive measure with a ruler of the height of the water level. For each scheme five groups of two pupils interacting together were interviewed.

	1	2	3	4	5
QUALITATIVE	Plotting the set of points using the kit	Prediction via sketching of the set of points for a narrower becher	Plotting with the kit to check	Prediction with a conical-shaped flask	Checking
QUANTITATIVE	Plotting from successive reading		Plotting through successive measurements to check		Checking

Figure 2 : Structure of the séquences didactiques

To analyse how the notion had developed or how it could develop further we use a micro-computer connected to a bit pad (also called drawing tablet). What is drawn on the tablet (12"x12") appears on the screen. Four phenomena were presented dynamically on the monitor. The first two showed a column of liquid going up, one steadily, another one at a faster rate from half-time onwards. The last two showed small rectangular spots appearing on the screen as a population filling an area. The patterns of growth matched those of the first phenomenon.

Water 1 - Sketch - Water 2 - Sketch - Population 1 - Computer - Population 2
no verification shows curve

Note: The pupil can ask for repetitions of the animation. It is provided with the correct graph only after "Population 1".

Figure 3: Structure of the testing procedure

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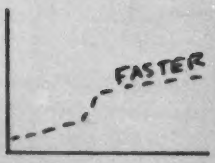
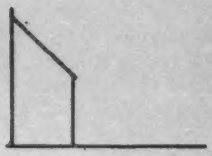
. Higher-quicker:
One student identified a curve put higher as denoting a growing faster.



. Fechner's law:
With the population growth we noticed Fechner reactions namely that increases were measure in relation to the "local value of the variables". In fact, for a steady growth, population seems to come in more than when an area is almost all packed up.

. Graphical reconstruction:
Several "personal" graphical interpretation was provided. The most interesting ones were ones were the curve was plotted from bottom to top the abscissa showing the variation.



Conclusions

Results do not suggest any particular approach since we think that a blend of both would be ideal. What can be derrived from a teaching perspective is how to conduct séquences didactiques to make them more successful.

The author wants to express his gratitude to his colleagues L. Charbonneau and Sophie René de Cotret for their collaboration in the project.

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RESULTS

The results are not described by right or wrong procedures. Indeed, the two schemes were meant to conduce to different performance which we wanted to analyse. With the qualitative scheme, time and motion were the main ingredients of the situations the pupils had to work on. They had in a way to "stop" the time and inject some measure. As for the qualitative scheme, the sequence of numeric increases could be easily obtained and a mastery of the idea involved integrating the elements of the sequence using time.

The table next page described the performance of each group. The number of groups we have does not allow any generalisation. However, specific protocols reveal a lot on the construction of the idea. They will be presented at the conference.

P: parallel responses C: correct (although diverse) L: linear
All graphs were made with points except when drawing presented or when indicated.

Figure 4: Code used in Table 1

Summary of main strategies

. Parrallelism:
Predictions for the curve of the narrower becher turned out to be a parallel over the first becher. This turned out to be a graphical equivalent to the additive responses to problems involving ratio (KARPLUS, HART...).

. Closer-faster:
In their predictions several pupils put the points closer to "mean" faster rather than adjusting the gradient.

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TABLE 1

		BOTTLE		GRAPHICAL PAD				
	DECREA	TESTING	CONICAL-SHAPED FLASK	G1 (water)	G2 (water)	POP. 1	TEST	POP. 2
QUALITATIVE	1 P		L	C	1	Photo	Symmetry	Photo
	2 P	-	Not defined	Point Closer→faster	1		ok	Point Closer→faster
	3 C^1	-		C	C	C	ok	C^1 C (little change in gradient)
	4 C	-	L (slightly bent)	C	C	C (Bruchner effect)	ok	C
	5 C	-	L (slightly bent)	C	C	C	ok	C
QUANTITATIVE	6 P	No	L (measuring)	Speed of pendil: speed of water		photo	1	photo
	7 P	IMPOSSIBLE SUPERIMPOSED	L (first one measured)	Points closer→faster	Points closer→faster		ok	Point closer→faster
	8 P	IMPOSSIBLE SUPERIMPOSED	L (slightly bent)	C	C	Rate of appearance (so many a second)	ok	Rate of appearance
	9 P	DONE	C	Pissorial end		Photo		
	5 P [illegible]	DONE	C (for one pupil) last point plotted, the rest adjusted	Double "done not stop"	higher→quicker	higher→quicker	Not Done	higher→quicker

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USING LITERAL SYMBOLS TO REPRESENT HIDDEN QUANTITIES

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Prior research has shown that beginning algebra students face major cognitive obstacles in constructing meaning for algebraic expressions. The research reported here is part of a teaching experiment whose objective was to develop and test a new approach in the introduction of this topic. This new approach presented algebraic expressions as answers to easily visualizable problems such as the quantity of dots in a rectangular array, the length of a line divided into segments, and the area of a rectangle. In all these problems, one of the dimensions was hidden and had to be represented first by a box used as a placeholder and then by a letter.

The research reported here is part of a teaching experiment whose prime objective was the construction of meaning for algebraic expressions. A review of current textbooks used in secondary schools confirmed that most introductions to algebra started with algebraic expressions and an analysis of their presentations indicated how difficult it was to introduce these expressions in a way which might be meaningful to beginning algebra students.

An alternate approach was investigated through the development and clinical experimentation of a teaching outline which sought to identify algebraic expressions as "answers to problems". The elaboration of this teaching outline took into account cognitive obstacles uncovered in prior research such as the lack of a numerical referent ("How can you multiply by x when you don't know what number it is") identified by Davis (1975) and the potential problems in-

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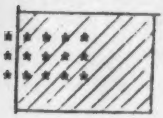
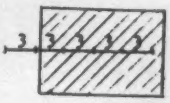
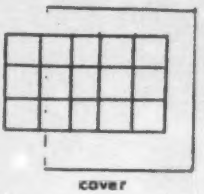
volved with concatenation when viewed by students in an arithmetic frame of reference (Matz, 1979). The introduction of algebraic expressions as "answers to problems" was expected to overcome the "name-process dilemma by which an expression such as $x+3$ represents both the process of adding 3 to x and also names the result (Davis, 1975), as well as overcome the student's possible inability to accept the lack of closure of such expressions (Collis, 1974).

In preparing the teaching outline, the type of problems to be selected had to be relatively easy so that "the answer" involved only concepts familiar to the students. In running the teaching outline through some pilot studies, it was found that subjects who had no difficulty in determining the area of a rectangle could accept the convention of using a letter to represent one of its dimensions and could even write as requested "area = 3 x a ". However, after 20 minutes of such work indicating great facility with the problems, a chance remark such as "How can you multiply a letter by a number ?" would destroy all our inferences about the construction of meaning by the subject. Such preliminary results confirmed those of Davis and suggested a more cautious approach. Rather than start introducing problems in which letters were to represent a specific unknown quantity (such as the length of the base of a rectangle), an intermediate step involving the notion of hidden quantity was to be used.

For our teaching experiment three problem types were selected on the basis of their easy geometric representation. The first problem type involved rectangular arrays of dots for which only one dimension (a row or a column) was shown, the second type of problem was that of a line segment in which the number of parts was hidden, the third problem type was that of the area of a rectangle with only one dimension shown. In all cases, the hiding was done by using a cardboard cover.







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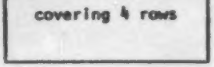
In our teaching experiment our subjects were 3 grade 6 students and 3 grade 7 students who had no prior introduction to a formal course in algebra. They were selected by the school authorities on the basis of their scholastic performance in order to provide us with a weak, an average, and a strong pupil from each of these grades. The students were pretested to ascertain that they could handle the three types of problems at the numerical level and that they perceived their multiplicative nature. Each student was taught individually and each lesson recorded and transcribed completely.

USING THE PLACEHOLDER BOX TO REPRESENT A HIDDEN QUANTITY

In order to make the introduction to algebraic expressions as simple as possible subjects were initially taught to use the familiar box as a placeholder for the partially hidden dimension. At first they were presented with a 5x7 array of dots in which only the first row was exposed: * * * * * * *



Here is a card. You can see a row of 7 dots. And I have hidden more rows, each with 7 dots. Here is the problem. How would I write the total number of dots if I don't know the exact number of rows ? So let me show you how I do it:

Number of rows = 7 x ☐

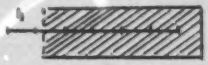
Since I don't know the exact number of rows, I am meanwhile using a box. Now I am going to let you have a peak at what I have covered and ask you to fill the right number in the box.

The use of the box did not appear troublesome for any of the students, except initially for Wendy, our average grade 6 student, who felt the expression was incomplete and had to provide a numerical value for the box upon seeing it. All 6 subjects correctly wrote the number 5 inside the box, thus indicating that they correctly perceived the box as representing the total number of rows, not just those that were covered. Given another such dot problem, all students were able to complete the statement "Number of dots = " by writing 8 x ☐ Finally, they were given an expression and asked to generate a dot problem:

Now it is your turn. Can you make up a dot problem like we just did, where 5 x ☐ is the total number of dots ?

A typical response was that of Gail, our strong grade 7 student, who hid her work from the interviewer and when she turned around, showed a row of 5 dots while covering the remaining rows. The other five subjects behaved in a similar manner thus indicating that they had grasped the intended meaning of the box as representing an undetermined quantity which was partially hidden. Students were then presented with a line problem:

Now look at this line problem. Each part is 4 units long. But you don't know how many parts there are. Can you complete this equation ?



covering 5 parts

Length =

(Uncover) So what number has to go into the box ?

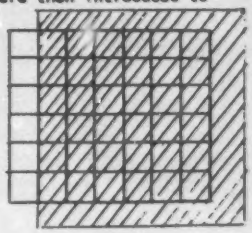
Five of the six subjects were able to transfer spontaneously their use of the box to represent the number of parts. Only Wendy needed to be reminded about the possibility of using it. The second line problem required a reversal:

Now it is your turn to make up a line problem where $3 \times \square$ is the length of the line.

Five of the six subjects easily generated an appropriate line problem. Only Filippo, our average grade 7 student, experienced some difficulty. He initially drew a line divided into 3 segments, each one unit long stating "I thought you meant there were 3 spaces and I had to find out what was in the middle space." Thus the reversal problem was valuable in uncovering and then correcting his misinterpretation. Students were then introduced to the following area problem:

Now I will draw an area problem in which I will only show you the height, and I will cover the rest of the rectangle. Can you complete this equation? Area = ?

(Uncover) So what number has to go into the box?


cover

All 6 subjects responded with the expression $6 \times \square$ to the area question. Each student then made up his own area problem. If this first part has been reported in such detail, it is because an entirely analogous line of questions were presented in the next part of the lesson which introduced the use of letters.

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THE TRANSITION FROM BOXES TO LETTERS

Following this initial work with the box as a placeholder, the three types of problems were re-introduced with the only change being a request that the student use a letter of his choice (instead of a box) to represent the hidden quantity. This request was motivated by the need to convey the arbitrary nature of the symbol, as had been shown by Wagner (1981). The first problem was presented as follows:

> Let's do another dot problem. You can see that I have a row of 5 dots, but you can't see the exact number of rows. This time I am going to use a letter instead of a box for the number of rows. Choose a letter. So let me show you how I write the total number of dots using your letter: Number of dots = 5 x a (or whatever letter is chosen).
>
> ☆ ☆ ☆ ☆ ☆
> covering
> 3 rows
>
> How I will let you look at the number of rows. How many rows are there? What does your letter stand for? Can you complete this equation?
> Number of dots = 5 x ?

All students appeared to accept the use of the letter they chose as part of the algebraic expression representing the total number of dots. As before with the boxes, they exhibited no difficulty in understanding that the letter stood for the total number of rows. This was even true for Wendy who in the pre-test had maintained that the value of a letter is determined by its rank in the alphabet (Chalouh & Herscovics, 1983).

The ease with which all the students went through the whole sequence of dot, line, and area problems seemed to indicate that they did not experience any cognitive dilemma in their use of letters representing hidden quantities and in the writing of algebraic expressions. The problem evidenced in the initial pilot study ("How can you multiply a number by a letter?") did not surface at any point in this lesson. The presentation used here provided a situation where the letter became a natural extension of the students' use of a placeholder, the box, and both were explicitly linked to a numerical referent.

Although all students had sailed through this first lesson easily, the newly acquired knowledge proved to be somewhat tenuous for three of our subjects. This became apparent in the homework they submitted two days later. They had

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been asked to make up: a dot problem where the answer is 3xa, a line problem where the answer is 5xb, an area problem where the answer is 6xd.

Three of the six students, Antoinetta (gr.6, strong), Frankie (gr.6, weak) and Filippo (gr.7, average), indicated some cognitive difficulty. Antoinetta did not bother to cover any dimension in the dot problem she had prepared; Frankie felt he had to ask "Should I cover it up ?" referring to his array of dots; Filippo drew a rectangle showing the height to be d units and the base to be 6 units long and wrote Area = 24, instead of 6xd, thus avoiding the use of the letter d for the presumably unknown quantity. For Antoinetta, it may have been a question of remembering how to set up these problems while for Frankie it could have been just a question of reassurance. However, the cognitive difficulties should not be judged as serious since they were easily overcome during the course of the review, as indicated by the success of these students with the remaining two problems of the homework.

BY WAY OF CONCLUSION

The final section of lesson I dealt with questions aimed at determining the students' perceptions of the differences and similarities between the use of letters and the use of boxes. They were asked: Do you find it more difficult to use letters than boxes ? Can you tell me why? The responses indicate that two of the subjects, Gail and Filippo, had more difficulty in using letters than in using boxes, each one highlighting possible confusions which may arise with letters. Gail pointed out that the use of different letters could imply perhaps different numbers resulting in a possible "mix-up". Filippo astutely brought attention to the dilemma that could be experienced with the letter x, whether it stood for multiplication or represented a number.

The next questions aimed at verifying explicitly whether or not the cognitive dilemma associated with the use of letters and numbers was present for any of the subjects by the end of our first lesson. They were asked: Does it bother you to see a number multiplying a letter, like in 3xa ? Can you tell me why ? What about boxes, does it bother you the same way when you see 3 x □ ?
All six subjects stated that it did not bother them to see a number multiplying a letter, thus confirming that no cognitive dilemma existed for them and suggesting that the primary aim of lesson I had been achieved. We thought that

the students were now ready for the next use of letters, that of representing a specific unknown quantity. The results are reported in a companion paper by Chalouh & Herscovics (1984).

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FROM LETTER REPRESENTING A HIDDEN QUANTITY TO LETTER REPRESENTING AN UNKNOWN QUANTITY

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Our prior success in introducing students to the use of letters as representing hidden quantities led us to believe that the transition to their use as representing unknown quantities would proceed without any major cognitive difficulty. In fact, all six subjects indicated that they were experiencing some cognitive discontinuity in this transition. This led us to re-assess our impression that they were developing an acceptance for the lack of closure of algebraic expressions. Although this cognitive discontinuity was easily overcome, nevertheless it brought out the complexity of the name-process dilemma.

In an companion paper (Herscovics & Chalouh, 1984) we have shown how literal symbols were used to represent hidden quantities in our three problem types. Since our subjects could now use literal symbols without experiencing any cognitive difficulties in the context of a hidden quantity, they were ready for Lesson 2 which introduced the use of the literal symbol as representing an unknown quantity. The problem types being the same as in the previous lesson, it was presumed that the use of a letter representing an unknown would be just an easy extension of its previous use in representing a hidden quantity.

Our success with the students in the first lesson, and, the students' success with the homework problems based on that lesson, led us to conclude that our subjects were beginning to accept the lack of closure of algebraic expressions, that is, the ability to hold unevaluated expressions in suspension (Collis, 1974). Thus we believed the subjects would have no difficulty in answering the first question in the second lesson, "Do you think you could write down the area of this rectangle?"

, since all that was

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required was for the student to write '8 x a'. It should be noted that the form of this question differed from the area problems in Lesson I, where each question was started by requesting the student to complete a statement such as "Area of Rectangle = ".

The responses given by our subjects to this first problem highlighted some unforeseen cognitive difficulties and revealed that the assumption that our subjects were beginning to develop acceptance of lack of closure was premature.

The subjects responded in the following manner:

WENDY (Grade 6 Student of Average Mathematical Ability)

Wendy first requested the use of a ruler to measure the length of a, the unknown dimension. Wendy's need to measure the length of a rather than provide an algebraic expression as the answer, could be partially explained in terms of the change in wording of the question mentioned above. However, it may also reflect the change in the connotation of the literal symbol (from hidden to unknown dimension). It may be easier to hold in suspension an algebraic expression such as '8 x a' in the context of a hidden quantity, for the hiding implies a second phase, that of uncovering, which will allow the student to evaluate the expression. On the other hand, in the context of an unknown dimension, the notion of hiding is missing and thus cannot help in holding '8 x a' in suspension. Thus in order to provide what she believed to be an answer in this context, she spontaneously asked, "Can I use a ruler?"

When she was requested to provide a response without measuring, she then wrote '8 x a= " and '8 x __ = ', and stated "I couldn't, I would have to find what replaces a." This response indicates the strong link of the letter with the placeholder interpretation. It should be noted that she wrote the equal symbol, thus indicating that either form of the expression was still incomplete. And it would remain incomplete until she would be able to evaluate a, as she stated, "I would have to find what replaces a."

It is only when Wendy was asked to complete the sentence, "Area =" that she

seemed to accept '8 x a' without feeling she had to evaluate a. Her expressed desire to find what replaces a is a clear indication of her inability to accept the lack of closure of the expression '8 x a'. The problem here is not limited to the lack of closure of the expression, but it also involves the "name-process dilemma". (Davis, 1975) This is very clearly illustrated by the fact that Wendy had no difficulty in completing the statement "Area =" and by her further comment indicating that the expression '8 x a' was acceptable only in this context ("unless you write 8 times a is the area"). A possible explanation here is that in the context of completing the equation "Area=" the expression '8 x a' need not play a double role, that of name and that of process. The word "Area" appearing on the left expresses the name, thus '8 x a' can be limited to express the process. It seems that the question "Can you complete 'Area = ' " might avoid the cognitive problems involving lack of closure and the name-process dilemma.

FRANKIE (Grade 6 Student of Low Mathematical Ability)

Frankie, unlike Wendy was able to spontaneously write '8 x a'. However, he demonstrated that he sensed it was not the answer by stating "You would have to find the number for a...you could measure it." Thus, in the sense that the algebraic expression is not the answer, he was unable to accept the lack of closure. As in the case of Wendy, the context of an equation "Area = 8 x a" appeared to have assisted him, and he no longer appeared to require the measurement of a. Thus within this context the answer seemed complete, since no desire for measurement was stated by Frankie.

ANTOINETTA (Grade 6 Student of High Mathematical Ability)

Antoinetta was initially totally unable to provide any response at all. And this was quite surprising since the pretest showed she had no problem with area in a numerical context; the area problems presented in Lesson I were handled easily with the literal symbol representing hidden quantities; and the first homework assignment was dealt with efficiently. So it was quite surprising to find that Antoinetta could not make the transition from letter used as hidden quantity to letter used as unknown quantity. In fact her paralysis indicates quite a major cognitive gap preventing this transition.

When probed about the meaning of the letter a, she responded numerically by estimating the length of the base. Reminding her of her previous work did not elicit any response. She had to be literally fed the answer to the problem by having her complete the equation "Area = 8 x ".

GAIL (Grade 7 Student of High Mathematical Ability)

Gail initially behaved similarly to Wendy, and provided at first the response "8 x a=", but then substituted the number 10 for the a (an estimate of the length of the base), arriving at a final numerical answer (8 x 10 = 80). Thus as with Wendy, the expression '8 x a' was incomplete to her, as indicated by her writing an equal sign after it, and by her further attempt at providing an estimated answer for the area. This behaviour strongly indicates her inability to accept the lack of closure of the expression '8 x a'. However, within the context of the formula, "Area = Length of Height x Length of Base" the substitutions of a numeral and a letter seemed acceptable since she wrote '8 x a' underneath 'Length of Height x Length of Base' without any hesitation.

FILIPPO (Grade 7 Student of Average Mathematical Ability)

As in the case of Antoinetta, Filippo could not respond to the initial question asking him to write the area of the given rectangle. And the reason appeared to be his numerical frame of mind which was indicated by his need to estimate the length of the base.

Within the context of the incomplete statement "Area = 8 x ", although he completed it with an a, this correct answer did not suggest to Filippo that he had finally arrived at a response to the original question. He clearly stated "If you are trying to find the answer you won't get it." Thus the algebraic expression to Filippo is definitely not the answer, even within the context of the equation. To Filippo, the expression is a statement of the operation to be performed to get the answer, as he stated "So you are saying you just have to say what you have to multiply to get the area." Filippo's difficulty to perceive the algebraic expression as the answer clearly demonstrates the Davis "name-process" dilemma.

YVETTE (Grade 7 Student of Low Mathematical Ability

Yvette had the advantage of a variation in the wording of the initial question. The interviewer immediately provided the incomplete statement "Area = ". This can explain why Yvette did not have the initial cognitive problems that the other students experienced, since she easily completed the statement with the expression '8 x a'. However, she then insisted that '8 x 2' and not '8 x a' answered the question "to write the area" which indicates that she also felt an incompleteness with the algebraic expression, even within the context of an equation. This need to provide a numerical answer shows her difficulty in accepting the lack of closure of the expression '8 x a'. She stated however, that if no final numerical answer was required, "you would just put 8 times a".

SUMMARY

Based on our success with the first lesson, where the letter was introduced as representing a hidden quantity, we did not anticipate any difficulty with the question "Can you write down the area of this rectangle, ? ?" since all that was required was for the student to write '8 x a'. However, the responses of each of our six subjects indicated that they all experienced new difficulties.

A first observation was that all students misinterpreted the question in the sense that they felt they had to provide a numerical answer to the problem. This need to provide a numerical answer did not manifest itself in Lesson I, where the literal symbol represented a hidden quantity. Thus it seems that this transition to letter used as unknown involves greater cognitive obstacles. The conjecture provided in the analysis of Wendy's interview seems to be a reasonable explanation: It may be easier to hold in suspension an algebraic expression such as '8 x a' in the context of a hidden quantity, for the hiding implies a second phase, that of uncovering, which will allow the student to evaluate the expression. On the other hand, in the context of an unknown dimension the notion of hiding is missing and thus cannot help in holding '8 x a' in suspension.

Of course, one can hardly discuss holding an expression in suspension without relating it to the acceptance of the lack of closure. Two of our subjects, Wendy and Gail, indicated that they viewed the expression '8 x a' to be incomplete by writing an equal sign after the expression. Both Frankie and Yvette wrote the expression '8 x a' but immediately made some reference to evaluating the a. Antoinetta and Filippo were not able to write the expression, whereas they had managed to do so in the previous lesson. Thus it appears our assumption based on the results of Lesson I were premature and that in fact our subjects did not accept the lack of closure of an algebraic expression.

While our subjects gave strong indications that they could not accept the algebraic expression by itself, they also indicated that it was far more acceptable to them in the context of completing the statement, "Area = ". A possible explanation of the students' greater acceptances of the expression within the framework of an equation can be sought in terms of the name-process dilemma. By completing the equation "Area = ", the left-hand side clearly expresses the name, thus leaving to the right-hand side the function of expressing the process, '8 x a'. Yvette indicated this process interpretation of the expression by responding to the question about writing the area of the rectangle without a final number answer: "You would just put 8 times a". But it is Filippo who expresses this idea most explicitly in commenting on the '8 x a' in the question, 'Area = 8 x a': "So you are saying you just have to say what you have to multiply to get the area."

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THE ROLE OF COGNITIVE MODEL BUILDING: A CASE STUDY

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ABSTRACT

Research into student understanding of algebraic variables serves as a case study from which an analysis is made of the role of explicit cognitive model building in four essential aspects of research, including methodology, data, scope, and applicability. The clear findings are the subsidiary role of model building in the construction of explanatory theory and the varieties of uses that the model building serves in this role. Its role in setting the other research parameters varies and does not allow simple characterization.

We examine a given, fixed research problem with a view to identifying in concrete terms the role of explicit cognitive modeling in four aspects of that research: (i) Choice of Methodologies; (ii) The type and richness of Data that is likely to be accomodated; (iii) The determination of the Scope of the research issues to be addressed; and (iv) The kinds of practical Applications expected from and provided by the research. The case study illustrates that the major identifiable role of cognitive modeling, however, is not its role in setting values of these four research parameters, but rather its central subsidiary role in supporting explanation, which is the primary regulating force in the research activity examined. Cognitive models assist in the building of explanatory theories, but do not themselves constitute theories.

We regard the special issues raised by the use of formal models as separable from those that we wish to discuss, so we consider only informal descriptive models and not simulation models (Weizenbaum, 1976). For a discussion of formal models, see (Lin, 1979; Kaput, 1984).

A CASE STUDY - RESEARCH ON THE MEANING OF ALGEBRAIC LITERALS

The research problem is the study of student interpretation of algebraic literals and the associated problem of student understanding of algebraic variables spawned by the now familiar "Students-Professors" translation task. The work began with the discovery of a frequent and surprisingly robust reversal error in a series of diagnostic tests given to incoming freshmen at a major public university (Clement, et al, 1981; Clement, 1982).

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(Similar phenomena were described at about the same time in England by other researchers but will not be reviewed here.) The following two-variable translation task produced error rates of 35-60%.

Write an equation using the letters S and P to represent the following statement: There are six times as many students as there are professors at this university. Use S for the number of students and P for the number of professors.

About two thirds of the errors were "6S=P reversals," a result which has held up in many replications across different populations. An immediate traditional mathematics education response to this sort of finding is to note that since such a reversed equation makes sense as an abbreviated English translation, one must teach against this tendency, e.g., caution against literal translations, remind students that algebraic letters stand for numbers, foster checking via numerical substitutions, and so on. In fact such (obviously sound advice) is implicit in some of the earlier understanding-of-variables literature and has appeared recently (Wollman, 1983). Another traditional response is to suggest that the students are exhibiting a weak understanding of mathematical variables, which understanding, given the strength and ubiquity of natural language as a means of representing ideas, is overridden by their natural language habits.

The cognitive modeling response exemplified by Clement (1982) and based on earlier work by Witz (1971) and Easley (1979) instead turned to a more detailed specification of the mental processes that would help differentiate correct from incorrect performance. But the clinical interview methodology used to uncover those processes simultaneously exposed the robustness of the error and its resilience to corrective instruction, thereby increasing pressure for "explanation," the response to which was heightened interest in building more complete cognitive models - the kind of positive explanatory feedback loop that likely motivates scientific research in any field. The key feature in this process, however, is the role of explanation as a driving force. The goal of traditional mathematics education research, curricular improvement, is not herein replaced as a fundamental goal, but instead, has increasingly explicit explanation as new vehicle for its achievement. Since pedagogical application was seen as waiting upon full explanation, curriculum development took the form of a short term tutorial strategy whose main effect was to confirm the depth of the pedagogical difficulty (Rosnick & Clement, 1980), enhance the drive for explanation, and push back the expected curricular payoff because the repair was now seen to involve mathematical topics stretching back through the algebra curriculum (Rosnick, 1983) and even earlier (Kaput & Sims-Knight, 1984).

A deliberately cyclical strategy developed whereby phenomena

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identified in clinical interviews were "confirmed" by their appearance in paper-pencil versions of the tasks administered to larger groups and then further subjected to detailed modeling of individual performances. In this way the verbal translation task was found to be a special case of more general translation difficulties involving translations of analogous quantitative relationships between other pairs of symbol systems, such as graphs, tables and so on (Clement, et al, 1981).

These initial cognitive models separated externally observable phenomena from hypothesized internal phenomena in an orderly way, using traditional language of mathematics education and elementary psychology on both sides of the separation. In this style Clement described two different incorrect approaches to the task, the superficial "syntactical translation" strategy and the more reasoned "static comparison" strategy (Clement, 1982, pp18-20). It is worth pointing out that this distinction, which appears to be new in the literature, resulted from the attempt to specify the underlying processes at a fairly high level of detail, because it is certainly not discernable from measurement of correct vs incorrect responses or even from the nature of the reversal error, which is identical for both erroneous approaches. Each of these strategies contrasted with the "active procedural" approach that was associated with correct performance. Identification of the procedural approach was then used as the base for another pedagogical strategy and further hypotheses, subsequently confirmed, founded on the inherently procedural character of computer representations (Clement, et al, 1980). Similarly, attention was directed to a contrast of the representation systems of algebra and geometry which utilized the translation task as a means to specifying further the properties of the representation systems themselves that seem to influence the ways in which mathematical ideas are encoded (Kaput, 1982). This latter work did not, however, have clinically based cognitive modeling as a primary outcome but rather used it as a tool to set up an experiment. The intention was to explore an additional dimension of explanation, both to shed light on the initial phenomena and to broaden the scope of the inquiry. This avenue of explanation is currently being pursued in a more general context (Kaput, 1983).

The apparent role of mental imagery in the semantic comparison approach to the original task and the appearance of analogous errors in picture to equation translations led to additional task analysis and clinical work that revealed the Students-Professors task to be near the middle of a continuum of like tasks with respect to the provocation of mental imagery in the representation of the given quantitative relationship. This led to additional cycles of clinical work and statistically analyzed tests based on manipulation of task variables designed to specify further the role of mental imagery and, in fact, verbal processing as well, in student conceptualization of quantitative

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relationships more generally (Sims-Knight & Kaput, 1983a). This latter work did not explicitly attempt to build cognitive models except to check for the presence of the appropriate processing, mainly because it was seen as preliminary to a search for more general translation processes that were hypothesized to lie at the base of a much wider collection of mathematical behaviors.

However, informal modeling had earlier helped identify such subprocesses associated with the original tasks as proportional reasoning, which in turn led to the identification of different sorts of error categories (on which the statistical analyses were actually done). Separate tasks associated with the subprocesses were then mixed with a wider collection of tasks, including "purer" translation tasks, with a view to isolating the elements of a "translation ability." Such general translation abilities were later indirectly identified using classic factor analytic methods (Sims-Knight & Kaput, 1983b; Kaput & Sims-Knight, 1984). There is, in this approach to "explaining" behavior, a presumed complementary relationship between subject abilities and task attributes. It is exactly at this point where cognitive model building would help verify this relationship on a cognitive process basis. For the most part, however, the explicit cognitive model building for the wider collection of tasks has yet to be done.

CONCLUSIONS

Although only a portion of the research in this domain was actually reviewed, and that in only a sketchy way, a few tentative conclusions may be drawn paralleling the four research issues broached in the opening paragraph. First, that the question of choosing to build cognitive models is not equivalent to a simple methodological question of clinical vs experimental methods should be clear, because each may be utilized, and in fact are both utilized in interactive ways in the case at hand. On the other hand it is probably safe to say that model building methodology tends to be focussed more on close observation of individuals than on statistical descriptions of populations. Second, the matter of data accomodation is complex, with cognitive models helping to find new data that in turn leads to research that is not cognitive modeling and which deals with entirely different forms of data (e.g., factor analytic descriptions of a subject population.) Nonetheless, the data deemed manageable, looked for, and hence found and described in a cognitive model building situation, tend to be rich and highly textured. Third, the cognitive model building can surely be credited with opening new avenues of approach, which in turn, may or may not require further cognitive modeling, depending on circumstances. However, the scope of research that tends to be addressed by model building, across time, populations, or even domains, is probably independent of the model building process, which can be set to virtually any level of detail or scope.

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Fourth, curricular implications may either be immediate as the result of modeling activities (by clearly specifying the differences between what can and what should happen in a student's mind), or they may be indirect as in the case examined. But because of the richness of accomodated data, and because the research statements tend to be "about" individuals (albeit modal individuals in some instances), who are, of course, the focus of teachers' attention, we can say that on balance conclusions drawn from explicit cognitive models tend to be more directly and recognizably applicable than other forms of research conclusions.

However, perhaps the most salient feature of the model building in the research cited is the variety of functions it performs in the service of explanation. But of course "explanation," especially psychological explanation, is a multidimensional, paradigm-relative term, not entirely independent of "description" (Suppe, 1974; Otorny, 1979). Unfortunately, space limitations defer a fuller discussion of the different ways that model building contributes to explanatory theory.

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A COMPARISON BETWEEN NOVICE AND MORE-EXPERT ALGEBRA STUDENTS ON TASKS DEALING WITH THE EQUIVALENCE OF EQUATIONS

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This paper reports on the equivalent-equations portion of a larger study on equation-solving. Six 12- and 13-year-old pre-algebra (novice) students were interviewed prior to taking part in a teaching experiment. Each was presented with three types of equivalent-equations tasks. The same tasks were also presented to nine more-experienced algebra students (14 - 17 years) in an interview setting. This paper compares the performance of both groups and suggests a model to account for the patterns of responses uncovered in the third type of task. There is also a brief report on how the teaching experiment on equation-solving indirectly altered the approaches used by the novices in the equivalent-equations tasks.

In algebra, all equation-solving transformations must maintain equivalence. In other words, each equation in an equation-solving chain must have the same solution as all the other equations in the chain. Most 12- and 13-year-olds entering secondary school have already had some experience with solving simple one-step equations of the form, $a + n = b$ or $a \times n = b$. However, this experience has focused primarily on finding the correct value of the letter n in these equations. Their elementary school experience has provided them with little, if any, explicit instruction in comparing the structural features of different arithmetic or algebraic expressions. Nevertheless, Chaiklin and Lesgold (1984) have shown that pre-algebra students do develop over the elementary years a set of implicit rules which they can apply to tasks involving the equivalence of numerical expressions. Since algebra can be considered as generalized arithmetic, it seems reasonable to assume that the ways of reasoning which students use in algebraic expressions and equations will be similar to the ones they use in arithmetic expressions. This paper reports on the kinds of reasoning used by a small group of pre-algebra students in determining the equivalence of algebraic equations. These ways of reasoning are then compared with the methods used by a group of more-experienced algebra students on the same set of tasks.

METHOD

A three-phase equation-solving study was carried out during the school year 1980-1981. The first phase involved interviews with 10 seventh graders (12½-13½ years) who had not studied algebra before; this phase was designed to uncover some of their pre-algebraic notions, in particular those on equations and equation-solving (Kieran, 1981). A subset of this pre-algebra group (6 children) was retained for the second phase of the study: a three-month teaching experiment on equation-solving. The solving method emphasized was that of performing the same operation on both sides of the equation. The teaching experiment included a pretest interview and two posttest interviews. Half of the questions of the pretest-posttests dealt with equation-solving tasks and the other half with equivalent-equations tasks. The third phase involved interviews with 9 more-experienced algebra students (they had all taken at least one algebra course) from grades 8 - 11 (6 from grade 8, 1 each from grades 9 to 11), to investigate their equation-solving procedures, errors, and methods of determining the equivalence of equation-pairs. This paper focuses on the parts of the second and third phases of the study which dealt with equivalent-equations tasks, in particular, on the pretest interviews with the pre-algebra students of the second phase and on the interviews with the algebra students of the third phase.

Three types of equivalent-equations tasks were presented to these subjects. These tasks were administered after a sequence of equation-solving tasks. All tasks were carried out in an interview setting; all interviews were audiotaped and transcribed in their entirety. Following are the three types of equivalent-equations tasks.

Type 1. Subjects were shown an equation written on a card and were asked what would happen if a certain number were added to one side. There were two of these questions.

i) "If $37x = 752$, what happens if I add 2 to the value of $37x$?"

ii) "If $x + 469 = 1351$, what happens if I add 10 to the left side?"

Type 2a: Subjects were shown an equation and were asked to do something to the equation to make the letter change/not change in value.

i) "Here is an equation, $3a + 156 = 372$. Can you think of something to

do to that equation so that x will change in value?"
ii) "Can you do something to $3a + 156 = 372$ so that a will not change in value?"
iii) "Here is another equation, $26c = 594$. Can you do something to that equation so that c will change in value?"
iv) "Can you do something so that c will not change in value?"

Type 2b: Subjects were then shown an equation printed on a card and were asked, "Can you make up an equation which will have the same solution as this one, without working it out?" There were three of these questions: i) $37 + b = 168$; ii) $28y = 362$; iii) $179 - t - 156 = 89$.

Type 3: Subjects were shown 7 sets of equations, one set at a time. Each set consisted of a model equation and several alternates. Each alternate was to be compared with the model. Subjects were presented with two equations at a time (a model and one of the alternates). The instructions for the task were as follows: "I will show you a card with an equation written on it in red ink. Then I will show you some other equations, one at a time, written in black ink. Without working out the equations, can you tell me which ones have the same solution as the red one? And why? Also, which equations do not have the same solution as the red one and why not? If you cannot tell, just say so." A sample set is:

Model Equation: $560 + b = 683$

Alternate Eqns.:
$560 = 683 + b$
$560 + b + 125 = 683 + 125$
$683 = 560 + b$
$683 + 560 = b$
$b + 560 + 125 = 683 - 125$
$b + 560 = 683$

These three types of tasks allowed us to investigate indirectly whether subjects could:

1. generate an equation equivalent to a given equation, using a specified transformation;
2. generate an equation which was i) not equivalent, then ii) equivalent to a given equation, when no specific transformation was suggested;

3. compare two given equations and specify the transformation which had been carried out on one of the equations to produce the other. Sometimes the two equations to be compared were equivalent; sometimes they were not. In giving a task like this, we hypothesized that we were going beyond the set of self-generated equivalent equations that a subject would use in equation-solving. Thus, we were attempting to uncover a wider range of techniques or procedures available to the subject for determining when two equations have the same solution.

RESULTS

Some of the subjects in the algebra group were not asked all of the questions of task types 1 and 2. These exclusions will be noted in the presentation of the results. The subjects in the pre-algebra group will sometimes be referred to as the novices; the subjects in the algebra group will correspondingly be referred to as the more-expert subjects.

TYPE 1 TASKS

Pre-Algebra Group: Three subjects correctly stated that if you do something to one side, you must do the same thing to the other side. Three incorrectly stated that if you add something to one side of an equation, you must subtract it from the other side. We have called this the Redistribution Error:

$$a + \underline{x} = b \Rightarrow a + \underline{x} + c = b - c$$

and have found that it occurred among novice equation-solvers with asymmetric solving preferences (see Kieran, 1983, for a discussion of symmetric and asymmetric solving procedures). The justification of this error seemed to be based on the following line of reasoning: Adding something on the left side makes the result on the right side larger; therefore, to restore the right hand result to what it was before, subtract that extra quantity from the result. It is an interesting fact that this error did not occur when subjects were actually solving equations. Furthermore, those subjects who clearly preferred to use symmetric solving methods in the pretest (eg, substitution, known number facts) never committed the redistribution error in the equivalent-equations tasks. There seemed to be a clear relationship between, on the one hand, preference for symmetric solving methods, awareness of the principle of performing the same operation on both sides,

bsence of the redistribution error, and, on the other hand, preference for symmetric (eg, inversing) solving methods, absence of awareness of the principle of performing the same operation on both sides, presence of the redistribution error.

lgebra Group: Only four of the more-expert subjects (grade 8) were given this et of questions. One committed the redistribution error; one used the rule of oing the same thing to both sides; one suggested adding the extra quantity on c the solution at the very end of the solving process; one suggested removing hat was added. Thus, their performance on this type of task was no better than hat of the novice subjects.

YPE 2a TASKS

re-Algebra Group: Five subjects could not respond to these questions; they did ot seem to know how to interpret them.

lgebra Group: Six grade 8 subjects were presented with this task. Four of them ere blocked by these questions. The remaining two used to a large extent the nversing transformation, i.e., transposing an element to the other side of the quation.

YPE 2b TASKS

re-Algebra Group: Three subjects said that they could not carry out the task instructed; they would have to solve the equations first. After solving, one these subjects suggested redistribution as an alternative to his inversing lving method, but abandoned the idea when he realized that the redistributed quation did not yield the same solution as the inversed equation. One of the ree remaining subjects offered inversed equations as her responses; another, quations with different letters; another, equations with one term increased in ze and another correspondingly decreased. This last suggestion was considered a variation of the Redistribution Error:

$$ax + b = c \Rightarrow (a - d)x + (b + d) = c$$

gebra Group: Four of the nine in this group of more experienced subjects also lieved that these questions could only be answered by first solving the given uations. The frequency of this response suggests that several of our more perienced subjects lacked the awareness that in every line of the solving procedure, the resulting equations are equivalent, that is, that each equation in e chain has the same solution.

TYPE 3 TASKS

Pre-Algebra Group: Space constraints do not permit a detailed presentation of all the results for this set of tasks. In general, subjects relied on the relative size of the numbers and on the numerical relationships represented by the equations to respond to this set of tasks. For example, they answered that certain equation pairs had the same solution because adding zero changes nothing, or because we can switch the order of adding, or because doing the same operation on both sides changes nothing. Similarly, they answered that certain pairs did not have the same solution because the left sides were not the same, or because the operation signs were different, or because the sum is always greater than either of the addends.

Algebra Group: The most common approach used by the more-expert subjects on this type of task was to transform one or both of the two equations by means of the inversing procedure until the two equations matched. If an inversing transformation could not produce a match, then subjects usually stated that the equations did not have the same solution and justified their replies by referring to specific numerical relationships or algebraic rules. It had been expected that they would try to establish which transformation had been carried out on the model equation to produce the alternate and then judge on the legality of such a transformation. However, they did not do this. They attempted to transform one or both equations in order to produce two matching equations.

A Model of Performance for the Type 3 Tasks

As with the other types of tasks, the novices made more mistakes than the more-expert subjects did. In fact, they replied incorrectly to the yes/no part of the Type 3 questions (Do the two equations have the same solution?) a total of 56 times out of a possible 179. This is in contrast to 27 (out of 261) incorrect yes/no answers on the part of the more-expert subjects. The novices' mistakes seemed to be due most often to failing to attend to all of the relevant distinguishing features of each equation in a given pair. In attempting to analyze what it was that the subjects, both novice and more-expert, tended to notice, and in what order, a model gradually took shape. The response patterns of the more-expert subjects to the third type of equivalent-equations tasks suggested

e following attentional model:
Pick out the differences between the two equations. Look at the numbers, then the signs, then the positions.
If the positioning does not match, will a switching rule produce a match?
If the numbers and/or signs do not match, will totaling or breaking into parts produce a match?
If signs and/or positions do not match, will inversing produce a match?

at is not captured in this model of more-expert processing of equation-pairs the parallel activity of accessing arithmetic and algebraic knowledge. Whenr they saw something in the two equations that did not match, they seemed to arch their relevant background knowledge for something which would confirm ir notion of the non-equivalence of the two equations, while they continued try to transform the two equations to create a match. Therefore, at any nt along the way, subjects could terminate the matching procedure, due to the cessful recall of a piece of appropriate background knowledge. Consequently, the Type 3 tasks, there seemed to be two levels of processing going on for more-expert subjects: a) one at the matching level, that is, trying to create atch out of 2 equations which had both external differences and similarities; one at the justification level, that is, trying to pick out of their mathemaal knowledge a reason why the two equations might not have the same solution.

novices, on the other hand, worked almost entirely at the justification level. y attempted to examine both equations, term by term, until a difference was nd. If they believed that the difference clearly prevented the two equations m having the same solution, they would stop immediately. If unsure, they tinued to compare both equations. They then decided on whether the differe(s) between the two equations was/were legal or not by reference to a set switching criteria, numerical relationships, or algebraic rules. The knowge base underlying these decisions differed for each novice. Some had strong es; others had weak ones; some held erroneous beliefs; others not. Some led on checks of numerical content; while others depended on rules only. In trast with the more-expert subjects, the novices seemed rarely to be able to a global picture of all the differences in both equations, unless the

ıations were very simple with only minor differences between them. In sum, the ɾices tended not to go beyond step 2 of the model representing more-expert ɾformance; furthermore, step 1 for the novices usually followed a left-to-ght search, rather than the pattern used by the more-expert subjects.

CONCLUDING REMARKS

can summarize the performance of the more-expert subjects on all of the equi-lent-equations tasks as being less expert than we expected. Their lack of owledge on how to construct an equation with the same solution as a given equa-on indicated little awareness that the solution of an equation remains the same each step of the equation-solving process. Despite their overall efficiency th asymmetric equation-solving procedures, ie, inversing, they had little if y knowledge of the equivalence of equations as it relates to equation-solving.

e performance of the novices on the equivalent-equations tasks changed drastic-ly on the posttest following the teaching experiment. Though we have not dis-ssed the teaching experiment in this paper, we should mention that it focused equations and equation-solving. More importantly, it did not include any olicit reference to the equivalence of equations; rather it emphasized the ecution of transformations which would not alter the value of the unknown, in rticular, the equation-solving transformation of performing the same operation both sides. The main effect of the teaching experiment on novices' handling the equivalent-equations tasks was a substantial increase in their use of is particular principle. Possession of the principle of performing the same eration on both sides eliminated to a very large extent the redistribution ror. Even though most of the novices used inversing to a certain extent in ving various equations, they tended not to use their knowledge of this pro-lure for the equivalent-equations tasks of the posttest. If they had used it e often, as the more-expert subjects had, they might have been able to avoid e of the erroneous conclusions they reached. Most of the novices seemed to largely unaware of the helpfulness of trying to match equation forms in the ivalent-equations tasks. They continued to rely on their knowledge of numer-l and operational relationships. The improvement in their first posttest

erformance was due to the acquisition of a principle which substantially expanded his knowledge base.

n fact, the results of the posttest suggest that the equivalent-equations tasks, ore than the equation-solving tasks, tapped the instruction that had occurred uring the teaching experiment. The novices used the principle of performing he same operation on both sides far more for the equivalent-equations tasks han for the equation-solving tasks. Thus, even in the posttest, the novices id not in general use their solving procedures in the equivalent-equations asks. This is in contrast to the more-expert subjects who relied strongly on heir equation-solving procedures for these tasks. Thus, the performance of the ovices in the June posttest (ie, the first posttest) did not mirror the performance of the more-expert subjects any more closely than it had in the pretest.

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THE LEARNING OF GEOMETRY AND MEASUREMENT

A FIRST APPROACH TO MEASURING (CHILDREN BETWEEN 11-13 YEARS OLD)[1]

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The purpose of this work is to make an analysis of what "measuring" means to children, studying the mental operations involved in the process, as well as the different stages of child's intellectual development that leads him to measure in a successful way. This is our first attempt to discover the obstacles and difficulties that a child encounters in a measurement task and what the teacher's attitude should be concerning such hindrances.

The methodology used in the research was analysis of a written test, direct observation of the activities developed in the classroom and analysis of written questionnaires designed for each activity.

Measuring is one of the fundamental topics in elementary mathematics since it is the main link with other sciences; moreover, it is a prerequisite for the study of other mathematical subjects. However, although most teachers recognize its importance, measuring is superficially treated in secondary school, and it is at this level that this topic will appear explicitly for the last time, ever, in any other curriculum whatsoever. The present work deals with the meaning of the measuring process to secondary school students and its purpose is to analyze the actions, the mental operation and the difficulties involved in this process.

Piaget and Inhelder [5] studied this topic and analyzed aspects such as: conservation, length and distance, area and volume. Later, the British study group, CSMS [4], carried out a research concerning these subjects with students between 11 and 16. Their results revealed that even at this level, one can find difficulties.

In the first part of our study, research was devoted to problem detection by means of a written test called "Diagnosis evaluation". The second part dealt with the design and development of activities seeking to solve some of the detected problems. Our sample were 52 secondary school students, ages 11-13.

DIAGNOSIS EVALUATION

Initial ideas for the items that conform the diagnosis evaluation were taken from the written test of the CSMS Mathematics Team's report [4]; nevertheless, our goals are slightly different. The aim of the CSMS study group was to develop a hierarchy of understanding in mathematics, whereas we were looking for a guiding principle to design measuring activities that could help the children overcome the detected difficulties in this field.

(1) The following persons assisted in this study: Professors L. Moreno and G. Zubieta; Postgrade students L.M. Bazarrechea, T. Bisalo, M.A. Fajardo, D.G. Fregoso, E. Galindo, M.H. García, M.T. López, J.C. Martínez, V. Padilla, M.C. Velázquez; all of them are members of Sección Matemática Educativa del C.I.E.A. del IPN.

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Our evaluation includes three parts: length, area and perimeter, and volume. In each part, the following aspects are analyzed: conservation, use of the ruler, use of units, and the use of fractions, plus the counting method or the use of formulae for finding area and volumes.

The test was applied before the start of the first secondary course. In the following paragraphs we outline the results obtained.

The sample we work with was small —52 students from two different groups; therefore, hierarchy was established by merely using percentages of correct answers (see Table 1). This hierarchy mostly coincides with the results reported by the CSMS group; yet some important differences appeared:

TABLE 1

Item	%Correct Answers	Item Characterization
2A	100	Comparison between lengths using the same unit.
7A	98	Area found by counting squares.
2C	94	Comparison between lengths, using the same unit.
5	92	Conservation of area.
10A	92	Area found by counting squares.
7C	90	Area found by counting squares.
6A	82	Conservation of area.
8A	81	Conservation of area.
8B	81	Conservation of area.
1C	79	Conservation of length.
3A	77	Use of the ruler.
3C	77	Use of the ruler.
14A	72	Volume found by counting cubes when all of them are not shown.
1A	70	Conservation of length.
14B	70	Volume found by counting cubes when all of them are not shown.
13A	68	Volume found by counting cubes when all of them are not shown.
11	66	Application of area formulae.
12A	64	Volume found by counting cubes when all of them are not shown.
9C	62	Volume found by counting cubes when all of them are not shown.
12B	62	Volume found by counting cubes when all of them are not shown.
1B	61	Conservation of length.
6B	61	Variation of perimeter with constant area.
3B	59	Use of the ruler.
12C	56	Conservation of volume.
7B	44	Finding area when the dimensions are fractional.
2B	42	Use of units.
10B	33	A fraction of the unit.
4	27	Perimeter of an irregular octagon.
9A	27	Conservation of perimeter.
7D	25	Finding area when the dimensions are fractional.
16A	24	Application of volume formulae.
14C	22	Application of volume formulae.
9B	18	Conservation of perimeter.
9C	16	Conservation of perimeter.
16B	9	Conservation of volume.
13b	3	A fraction of the unit.

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a) Items 1B, 6B, 3B, 12C, and 16B were classified by the CSMS group in level 1 (between 73 % and 89 % of correct answers). To our sample these items appeared to be more difficult: correct answers were 61% for 1B and 6B; 59 % for 3B; 55% for 12C; and only 9 % for 16B.
b) Items 9A, 9B, 9C related with conservation of perimeter turned out to be very difficult.

1. **CONSERVATION.** For items related with conservation, Table 1 shows a wide range of correct answers, with percentages as low as 9% for volume and as high as 92 % for area. This may suggest that there is a number of students who have not yet developed conservation completely.

It was suprising for us that items on conservation of area were, in general, easier than those about conservation of length. One was forced to ask himself whether the length concept would require higher abstraction than the area concept.

2. **USE OF THE RULER.** About one third of the group had problems with the use of the ruler, and about two thirds of the students had difficulties when a fraction appeared. This is something more than a mere lack of skill, since it implies that the measuring process (measuring length, in this case), as an iteration of equal intervals, has not quite been internalized and the task is done mechanically, without thinking.

3. **USE OF UNITS.** The measuring process cannot be grasped, unless the individual has developed a well established concept of unit. The lack of this concept is one of the measuring aspects that gave rise to the greatest number of problems, since it was found that more than half of the students used units incorrectly.

4. **FINDING AREAS AND VOLUMES.** Our results show that most of the children find areas and volumes mainly by counting units, although in some cases this method was much more complex than it would have been by the use of formulas. This is particularly true in the case of volume, because many children cannot visualize the units that are not shown. An important aspect was revealed here: spatial imagination is a scarcely developed ability among secondary school students.

5. **FRACTIONS.** Throughout the test, it could be observed that items become much more difficult when a fraction is introduced, either in the question or in the answer.

DESIGN AND EXPERIMENTATION OF LEARNING ACTIVITIES

Considering the results of the diagnosis evaluation, we carried out a structural analysis of the curricula design [1], to be used by teachers of the school in which the research was held. We designed sequences of learning activities related with length measurement and with the development of spatial imagination. In order to implement these activities, children were organized in small groups of three; the problems were posed to them, and their spontaneous reactions were observed, giving directions only when requested. Children had to record their measurements, they answered an individual questionnaire and later researchers and students together, discussed the

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results in the classroom. In every measurement activity, lengths to measure were equal or proportional, this was not the case with the units employed. This situation was designed to make the student think about the meaning of the resulting number in a measurement.

We will now briefly describe the learning activities and comment on some interesting observations.

MEASURING WITH RIBBONS. Children had to measure the perimeter of a basketball ground using only ribbons. Each group of students had one ribbon, which length they did not know. All the ribbons used were different in length. None of them was a multiple of the perimeter. The purposes of this learning activity were: i) to reaffirm the unit concept by emphasizing its arbitrary character against the idea that "the only valid measuring system is the decimal metric system"; ii) to become aware of the correct iteration of the unit in order to internalize this action as an intellectual operation; iii) to pose a situation in which the action of dividing the unit into fractions appears in a natural and necessary way; and iv) to appropriately register and operate the measuring results.

MEASURING CURVES. In the first part of this activity children had to measure the length of a curve, drawn on a piece of cardboard, using short, straight wooden sticks of different, non-specified lengths. In the second part, they were told to measure the same curve with a wire or a string. Then, they were to measure the stretched string with the help of the wooden sticks and compare this result with the results obtained in the first part. The purposes of this activity were: i) to reinforce the objectives of the preceding experience; ii) to introduce the idea of approximation, inherent to every measuring process; and iii) to explore whether children developed any ideas about curve rectification.

THE MEASURING WHEEL. Children had to measure two line segments drawn on the floor using a wooden disk. The length of one of the line segments was a multiple of the disk circumference, the other one was not. The circumferences of the disks used by the different groups were not equal. In this activity our purpose was to confront children with an unusual measuring situation in which neither the unit nor the tool were conventional.

The following are some of the most interesting reactions observed among children in the activities related with length measurement:

1) Most of the children showed mistrust in the use of units which were different from those of the decimal system.
2) Recording fractional results was a problem for almost every student; in some cases they used a different unit to express a fraction (i.e., they completed the measure with parts of their bodies, such as a handbreath, which name in Spanish is unfortunately related to one-fourth, and recorded the results as so many primary units plus so many "fourths").
3) In the discussion of the results obtained for the same length by different groups, there was a tendency to classify them as "good" or "bad" results, without doing any valid analysis thereof.
4) In the activity named "Measuring curves", it was evident that some of the children still had

problems with the conservation of length, since they tried anxiously to keep the string attached to the shape of the curve, and they resisted to stretch the string while they were measuring

DRAWING. Since the results of the diagnosis evaluation showed that children have serious difficulties with respect to the development of spatial imagination, we carried out an experiment to explore this ability among our students. We based our design on the experimentation reported by N.F. Chetverukhin [2], and called this activity "Drawing".

In this first part of this activity, children were asked to imagine and then to draw six objects: a brick (or a matchbox), a glass of water (not full), a table, a melon on a plate, a teapot, an airplane (or a bicycle). In the second part, a picture of a bottle of water (not full) was shown, and children were asked to draw the bottle imagining that it was at an angle. The purposes of this activity were: i) to observe how children use parallel perspective; ii) to observe what resources children use to represent solid objects; and iii) to observe how children use a reference system to locate a body in space.

Most of our children have great difficulties when it comes to graphical representation of solid bodies, and their drawings are obviously flat. In our curricula no special emphasis is made to develop spatial imagination.

The results outlined throughout this paper, allow us to assert that, in a general way, the fundamental measuring aspects have not yet been mastered at the begining of the secondary level, since there are still difficulties present which are related with length and area conservation. Also, at this level, a student does not have an adequate conceptual support: area and perimeter notions are often confused; the use of fractions presents many difficulties; the development of spatial imagination is poor; and a fixed measuring system is introduced far too early in the curriculum of elementary school, thus creating a barrier to the complete understanding of the unit concept.

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THE COMMUNICATION OF SPATIAL INFORMATION BY MEANS OF CODED ORTHOGONAL VIEWS[1]

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ABSTRACT

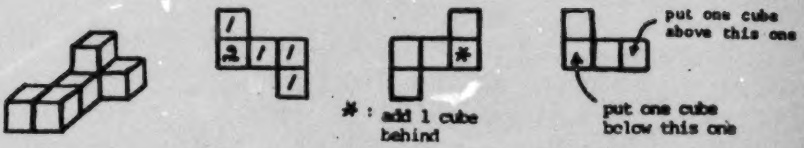
During our presentation, we are first going to describe the long-term research we have initiated this year. Its objectives are to study: (1) the nature and the development of the ability to communicate spatial information about polycubical shapes by means of what we call "coded orthogonal views"; (2) the influence of social interaction on the development of the preceding ability; (3) implications of the above for mathematics education.
Then, we will discuss the first phase of the research, namely the construction of an ordinal scale which will be used subsequently to assess the development of the above ability in subjects. Eight classes of students of various grades were administered a task and the results obtained were treated according to the scalogram-categorization technique, supplemented by common factor analysis. The resultant ordinal scale will be outlined during the oral presentation.

INTRODUCTION

During recent exploratory studies (Gaulin & Puchalska, 1983), it has been observed that when facing the task of representing three-dimensional polycubical shapes on paper, a great number of students of various ages spontaneously make use of what we call "coded orthogonal views" or COVs. Similar results have been obtained independently by Bessot & Eberhard (1982a; 1982b) and Franchi (1983). The three figures on the right below are examples of COVs represen-

(1) The research reported here is funded by a grant from "Fonds FCAC pour l'aide et le soutien à la recherche", Québec City.

ting the solid drawn in perspective on the left.



Any COV of a solid S consists of an orthogonal view of S, along with complementary information conveyed by means of a code, i.e. a structured system of signs and/or words complying with some conventions and rules of arrangement (Laborde, 1982). As can be seen from the above examples, each COV of S corresponds to an underlying mental structuring of S as well as to a more or less explicit type of coding strategy. Accordingly COVs are analytical graphical representations (as opposed to global ones like perspective drawings) and they constitute one particular type of "coded graphical representations" or CGRs, which include various kinds of drawings complemented with coded information.

OBJECTIVES OF THE RESEARCH

This year we have initiated a long-term investigation, which aims at studying: (1) the nature and the development of the ability to communicate (i.e. both encode and decode) spatial information about polycubical shapes by means of COVs; (2) the influence of social interaction on the development of the preceding ability; (3) implications of the above for mathematics education. At this stage, we found it desirable and more realistic to restrict our research to COVs (rather than studying CGRs in general) and to polycubical shapes (easily constructible with interlocking cubes).

The third objective is particularly important for us, considering the importance of developing students' spatial intuition, including their ability to visualize and to communicate information about three-dimensional geometrical shapes and relations. As emphasized by Caron-Pargue (1979, Ch. 1), perspective is just

one among many modes of graphical representation, each one having its characteristics and its merits, and we feel that in teaching students should be encouraged to use various types of plane representations, including COVs which are practically unknown in schools (except by a few students taking a course in technical drawing).

Accordingly our research is clearly interdisciplinary, since it has connections simultaneously with: semiotics; cognitive psychology; graphic information processing (see Bertin, 1981); social psychology; didactics of mathematics and of other disciplines like geography (where the processes of map making and map using are given some attention).

RATIONALE OF THE RESEARCH

We feel that this type of research is becoming growingly important, in particular for the following three reasons.

(1) The study should provide valuable information for the study of the (more general) ability to communicate spatial information by means of CGRs and consequently it should contribute to a deeper study of "graphicacy", which has been defined by Balchin (1972) as "the communication of spatial information that cannot be conveyed adequately by verbal or numerical means". Proficiency in graphicacy is considered by many as a fundamental goal of education for all:

> "Balchin and Coleman describe *literacy*, *numeracy* *articulacy* (subsequently superseded by *oracy*) and *graphicacy* as the four 'acies' or 'aces' for short. (...) Neither words nor numbers nor drawings are superior or inferior as modes of communication. They are only more suitable or less suitable for particular purposes, and each ranges from the very simple to the extremely complex. They complement each other and achieve their highest level of communication when properly integrated. (...) Balchin argued that the discussion ought to be more concerned with *modes*

of communication and that the three Rs should be replaced by the 'four aces'... (Boardman, 1983)

(2) The influence of social interaction on the development of cognitive abilities has been relatively little studied. It is only recently that researchers have proposed a *psychosociological approach* to intelligence, taking into account both the work of the Piagetian school and the contribution of social psychology (see Perret-Clermont, 1981; Doise & Mugny, 1981). (The way social interaction will be handled as a variable in our research will be described in another paper.)

(3) The study might have interesting implications for mathematics education. As explained already, it should give interesting clues about how to use COVs as a complementary means of developing students' spatial intuition, the way it has been attempted in a few projects (see Gaulin & al., 1983; IOMO, 1976). On the other hand, the influence of social interaction has recently become an important research concern in mathematics education, especially in France in the work of Guy Brousseau and others. It is certainly attractive to try to follow that promising trend.

FIRST PHASE OF THE RESEARCH: CONSTRUCTION OF AN ORDINAL SCALE

The first phase of our research was realized during Summer 1984. It involves the construction of an ordinal scale which is going to be used subsequently, e.g. to assess the influence of social interaction on the development of the ability to communicate spatial information by means of COVs. Such a scale purports to describe both (1) the degree of complexity of the task of representing on paper, by means of a COV, a given polycubical shape looked at orthogonally from a particular view; (2) levels of development of the ability to communicate spatial information about polycubical shapes by means of COVs.

At the time this paper was written, the construction of the scale was under way. In what follows, we are going to present only a sketchy description of the metho-

dology followed. As a first step, after careful analysis, a theoretical ordinal scale was postulated, with eight levels and representative shapes for each of the levels were constructed. As a second step, a task was administered individually to students of eight classes from grades 4, 5, 6, 7, 9 and 11. Each subject was given one set of eight shapes (built with plastic interlocked cubes) representing the levels postulated, and was instructed to do the following with each one successively: (1) Keep it in front of you, at the eye level, in the position indicated by a sticker on one face. (2) Observe the outline of the shape looked from that position. (3) Draw that outline on a given mm-squared sheet of paper. (4) Then add information by means of signs or words in order to get an adequate representation of the three-dimensional shape. To facilitate the understanding of the task by the students, the "outline" to be drawn was described as the shape of the smallest hole (in a vertical wall parallel to the face observed) that would allow the shape to go through the wall, and a few examples were discussed with the class before the subjects started to work individually.

As a third step, the results obtained were treated according to the scalogram-categorization technique, described by Noelting (1980a; 1980b), which was supplemented by common factor analysis to group items giving the same pattern of success and failure. [In a previous research concerning the development of the concepts of ratio and proportion, Noelting (1980a, p. 250) has found a large amount of convergence of those two methods, which appear to be complementary, one analyzing the data from a correlational point of view, the other in a structural perspective. The concept of level of development is accordingly set on a sounder basis.]

The resulting ordinal scale will be outlined during the oral presentation.

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INTERIOR STRUCTURINGS OF GEOMETRICAL OBJECTS: AN EXAMPLE OF INFRALOGICAL GROUPINGS

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ABSTRACT

Different types of cognitive structure, formulated in terms of groupings, are suggested to underly the first three levels of thinking according to van Hiele (Lunkenbein, 1983). In particular, infralogical groupings characterize an individual's reasoning on the first, the visual level of thinking. This presentation resumes the results of an experiment designed to provide some examples of infralogical groupings and to observe, in the individual's activities or expressions while carrying out a particular task, indications of the presence, the absence or the emergence of such groupings. Counting the faces of various polyhedra is the specific exercise suggested in this experiment. This task generates, in the mind of the individual who counts, a structural vision, an interior structuring of the geometrical object which decomposes the object into easily enumerable parts, the recomposition of which results in the total object. Examples of such interior structurings are given and observations as to how they are manifest in the individual's counting procedures or explanations are reported.

Groupings as Underlying Structures of van Hiele Levels of Thinking

Groupings as formulated by E. WITTMANN in 1975 and 1982 have been suggested as mental structural images underlying an individual's thinking pattern and characterizing as such his level of thinking in a given context according to van Hiele (LUNKENBEIN, 1983).

Generally, a grouping consists of

(1) "static" elements, called states or objects and which represent instances of a given conceptual context or problem situation;

(2) "dynamic" elements, called operations or transformations that act on the states or objects and relate them to each other;

(3) properties of or relations between states or objects brought out as the effect of the operations on the states.

According to the nature of their states and operations, groupings can describe rather different outlooks on a given conceptual context. Three types of groupings are formulated in order to specify the notion of underlying structure of the first three van Hiele levels of thinking.

Infralogical groupings, characterized by the spatio-temporal aspect of their states and operations, describe more precisely the underlying visual or spatial structures of van Hiele's first (visual) level of thinking. Groupings of the classes of a partition specify the underlying structures of the second (descriptive) level of thinking. Finally, groupings of logical inclusion are suggested to specify the underlying logical structures of van Hiele's third (logical) level of thinking.

While examples of the latter two types of groupings are readily available and quite easy to describe, examples of infralogical groupings are not abundant in the corresponding literature. This is probably due to the nature of this type of grouping.

The Nature of Infralogical Groupings

Infralogical groupings are the result of activities involving concrete materials and actions: spatial entities or objects are transformed, moved or decomposed actually or mentally, constituent parts or elements are identified and reconstituted into the more specifically organized same entity or into another related one. Such activities generate the conception of intrafigural relations and spatial properties of specific objects as well as interfigural relations and the differentiation of various spatial entities. So the states of infralogical groupings are spatial entities or objects as well as their constituent parts or elements; the operations of such groupings are transformations performed on these spatial entities and, even if done mentally, are directly derived from concrete manipulation; the effects of such activities are the conception of spatial organizations or structurings of geometrical entities or between geometrical objects giving rise to the recognition of relevant and differentiating properties.

Infralogical groupings are derived directly from concrete action. They are manifest through coherent action of the individual in a given problem situation and are described by its conceptor only through appropriate action and gesture and, if at all, through very rudimentary vocabulary.

This may be the reason for the lack of documentation of this type of grouping both concerning the description of examples and the observation of individual learners.

Interior Structurings of Geometrical Objects

As an example of an infralogical grouping let's consider a concrete object of the shape of a regular icosahedron. At first glance we might observe its

ball-like overall shape or have a rather static vision similar to its perspective drawing.

Figure 1: Static vision of the object

Certain concrete actions however, like turning the object around an axis passing through two opposite vertices, might bring out the vision of the object as being composed of two caps (of the shape of pentagonal pyramids) and a belt (of triangles).

Figure 2: Interior structuring of the object

In such a vision, the object is decomposed mentally into constituent parts which consequently can be recomposed into the original object. For the individual having had such experience, the given object has ceased to be just a ball-like global shape; it has been transformed, by an interior structuring, into a structural image identifying constituent parts and relating them into an organized whole.

Such interior structurings arise directly from concrete action; they enrich the individual's knowledge through a visual structure, a spatial organization of the geometrical object, which in turn gives rise to the recognition of relevant geometrical properties.

Interior Structurings through the Actions and Expressions of Individual Learners

The activity of counting the faces of polyhedra requires, on the part of the individual, an organization of his actions and the invention of a procedure so that he or she can be sure of the obtained result. Such a systematic procedure must be based, if no other aids are available, on a decomposition of the total number of faces in constituent and easily countable parts, the recomposition of which amount to the whole polyhedron. This means that such procedure is based on an interior structuring of the polyhedron.

In order to observe the conception of interior structurings in student's actions or expressions, we therefore asked them to count the number of faces of certain polyhedra and to show or to explain how they do it and how they can be sure of the obtained result. The results of this experiment are summarized here in terms of phenomena of the presence and the identification of interior structurings conceived by individual learners. These results are documented more explicitely in Lunkenbein, Allard, Goupille, 1981 and 1983.

The absence of interior structurings or the confusion of fragmentary structurings are manifest through a random, unorganized and often hesitating counting activity. Frequently the individual forgets to count some faces or counts others several times. In opposition to this, the presence of interior structurings is expressed in a systematic and well organized counting procedure, which can be reproduced if necessary. Counting is done without hesitation and the results are expressed with conviction.

Particular structurings can be identified through specific counting procedures. For example, the structuring of the rhombi-cuboctahedron in two opposite caps of 9 faces each and a belt of 8 squares is expressed by the counting and showing of the 9 cap-faces twice and seperately from the 8 belt-faces, which can be symbolized by the expressions 9 + 8 + 9 or 9 + 9 + 8 or 8 + 9 + 9.

Figure 3: Rhombi-cuboctahedron as two caps and a belt

On the other hand, the counting of four triplets of squares, four triplets of triangle-square-triangle and two single tops (4x3 + 4x3 + 1 + 1) indicates what we called the orange-structuring of the rhombi-cuboctahedron.

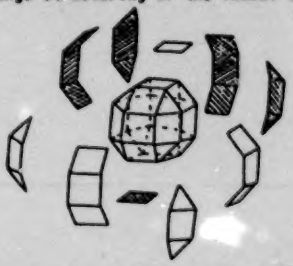


Figure 4: "Orange-structuring" of the rhombi-cuboctahedron

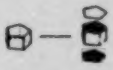
Different individuals may conceive different structurings of the same polyhedron and a particular individual may perceive one polyhedron in several different structurings.

In addition to these indications of the presence in the individual of particular interior structurings, it was most interesting to observe the progressive emergence and the assertion of such structurings. Confronted for the first time with the task of counting the faces of a more complexe polyhedron, most individuals proved to be rather perplexed and disorganized. Some of them, inspite of a presentiment of not being able to succeed, engaged immediately in the counting activity, resulting often in failure and causing repeated and varying attempts. Others, in the contrary, exposed more prudent behavior: they move and turn the object in their hands or on the table in order to find some way of organizing its different parts and to assure systematic counting.

In either case one witnesses the progressive emergence of a spatial organization becoming more and more articulate and relevant for the counting process. Although impossible to describe in detail the intense mental activity taking place during this period of emergence, the movement of the counting fingers, as well as the series of number words spelled out by the subject are revealing indicators for such activity. First of all, these observations seem to confirm that the actual manipulation of the object, be it during immediate counting attempts or be it during conscious observation in vue of a counting procedure, is the origin of any organization. Such manipulation is followed by attempts of systematic counting which in turn might reveal partial or instable structurings possibly disappearing again very quickly because of its incompletion or inappropriateness or because of the emergence of another competing structuring. Such competing structurings sometimes create conflicts since the individual might not be able to dissociate them consciously. Through repeated counting attempts some structurings are rejected while others are proven usefull and relevant for a counting procedure and will be retained.

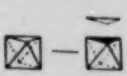
Some already conceived structurings which have proven appropriate for certain polyhedra are preserved by the subject and applied again in the consideration of new objects. This phenomenon of conservation was observed in the following sequence, where the individual first organized:

- the pentagonal prism as a pair of opposite faces connected by a belt of rectangles;



- then the pentagonal antiprism as a pair of opposite faces connected by a belt of triangles;

- then the regular octahedron as a triangular antiprism (usually this object is perceived as a double pyramid).



After that the subject tried to apply this structuring of two opposite faces connected by a belt of faces to the regular icosahedron for which such structuring is inappropriate. This perception had first to be discarded before another, more appropriate structuring could emerge.

Sometimes the application of a preconceived structuring would allow a student to succeed in counting the faces of a polyhedron which he could not do before. Such a situation was observed when a student first was unable to systematically count the faces of the regular icosahedron. After having structured the rhombi-cuboctahedron as two opposite caps and a belt of squares, he "discovered" the same structure in the regular icosahedron, which he then saw as two pyramidal opposite caps and a belt of triangles.

Conclusion

By these observations we not only are able to give some examples of infralogical groupings as they are manifest in the students actions and expressions, but we also learn about their emergence from concrete manipulation in connection with a given task or problem. The study of such groupings through the observation of their nature and their emergence as well as their development yielding in the conception of relevant properties of the objects in question hopefully will not only clarify the character of these structures underlying van Hiele's first level of thinking but also specify progression of learning on this first level towards the second level of thinking.

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VAN HIELE LEVELS OF THINKING OF SIXTH GRADERS

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ABSTRACT

This clinical study investigated the thinking in geometry of 16 sixth graders according to Dutch educator P. M. van Hiele's five levels of thinking. Subjects were interviewed for 6 - 8 45-minute sessions as they worked on instructional-assessment modules. Analysis of videotaped sessions indicate that students fall into three groups: level 0 thinkers who consistently deal with shapes by their appearance as a whole; students in transition to level 1 (characterizing shapes by their properties); students progressing toward level 2 (giving informal deductive arguments). Results suggest that the assessed level of thinking was affected by the student's prior learning in geometry, language ability, and experiences with activities during the clinical interviews.

This paper deals with research on the thinking in geometry of sixth graders. The conceptual framework for the study is model consisting of five levels of thought, proposed by the Dutch educators P.M. van Hiele and his late wife, Dina van Hiele-Geldof in 1957. This model has motivated considerable research and resultant changes in the geometry curriculum by Soviet educators during the 1960's (Pyshkalo, 1968; Wirszup, 1976). More recently, the model has been researched in the USA (Burger, 1982; Geddes et al., 1982; Usiskin, 1982).

According to the model, the learner, assisted by appropriate instructional experiences, passes through 5 levels of thinking: recognition of shapes by their appearance (level 0),

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nalysis of shapes in terms of their parts and characterization of hem by properties (level 1), logical ordering of properties and nformal proofs (level 2), deductive reasoning in an axiomatic eometry system (level 3) and finally a rigorous study of axiomatic eometries (level 4). Van Hiele claimed that the learner cannot chieve one level without passing through previous ones. Also, he tated that progress from one level to the next is more dependent on ducational experiences than on age or maturation, and that certain ypes of experiences can facilitate (or impede) progress within a evel and to a higher level. Also, concepts are experienced in ualitatively different ways at the various levels, and each level as its own language.

his paper is based on part of a 3-year study (Geddes et al.,1982) hich had four major research goals: (1) translation of important an Hiele sources into English and development of an operationalized ersion of the model in terms of specific student behaviors; (2) de-elopment of three instructional-assessment modules (Properties of uadrilaterals; Angle Sum of Polygons; Area) for use with subjects uring clinical interviews; (3) conduct of the clinical interviews ith sixth and ninth graders and analysis of the data; (4) analysis f geometry material in three major USA text series (grades 1-8) ccording to levels. Major questions of the clinical part of this tudy were: what type of thinking did students exhibit and at what evels?, did they progress within a level or to higher level as a esult of instruction?, and what difficulties did they encounter?

ixteen sixth graders (9 boys, 7 girls) were each interviewed for 6

45-minute sessions. All completed Module 1, some in 3 sessions the slowest in 7. Six students completed Modules 1,2, and most . A student's level of thinking was assessed from two points of as the student did the modules. The first was the student's ry level" relative to geometry topics in the modules. The nd was what might be termed the student's "potential level," ssed by examining the student's thinking during and after truction. Analysis of student thinking was based on examination videotaped sessions and related written assessment forms.

lts of the study indicate that these sixth graders fell roughly three groups. One group consisted of three level 0 thinkers began at level 0 and for the most part, remained at level 0, after instruction. Their thinking showed a lack of analysis of es in terms of their parts, lack of familiarity with basic metric concepts and terminology, and poor language (vocabulary grammar) both generally and in mathematics, especially ressive language. These students frequently forget terms and epts even shortly after they had been introduced by the rviewer. Two of the three were considerably below grade level achievement in reading and mathematics. All had a weak kground in school geometry and also had difficulty with thmetic concepts and skills. In fact, these three seemed to be metry deprived."

econd group had five students who began in level 0 but made gress within level 0 (learning basic concepts and terms) and into el 1 (using these concepts to describe shapes and to formulate

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properties of shapes, in particular, familiar ones such as squares, rectangles). However, they had difficulty characterizing less familiar shapes (e.g. parallelograms). Their progress was marked by instability between level 0 and level 1, and careful instruction and review of concepts and terms was needed to sustain their progress. One of these five was below grade level, the other four were on grade level. They tended to be more verbal than the three level 0 thinkers, but they had difficulty expressing themselves in standard geometric terms and often used manipulatives (e.g. D-stix) in checking properties. These students also seemed to have a weak background in geometry.

A third group consisted of eight students who began in level 1 but needed to review some concepts and to become more fluent with level 1 language for describing shapes in terms of properties. These students progressed toward level 2 by following and then summarizing arguments, for example why the opposite angles of a parallelogram are equal. A few progressed farther and began to give informal proofs more independently and with more detail. Initially, however, most students equated "proof" with generalization by examples. Only gradually, after experiencing some deductive explanations in Modules 2 and 3, did they seem to acquire a sensitivity to informal deduction. These students were at least 2 years above grade level in achievement. They were quite verbal and tended to express themselves confidently. They also seemed more reflective about the questions and problems in the modules and about their own thinking.

Results indicate that there was a wide range in levels of thinking

among sixth graders - from some who were consistent level 0 thinkers to some who were able to give informal deductive explanations (level 2). These results seem to conflict with those of Burger (1982) who found mainly level 0 thinking for subjects in grades K-8. However, Burger's assessment tasks involved entry level of thinking, not potential level as a result of instruction. In the this study the entry level of the sixth graders was mainly level 0 or at times level 1, much like Burger found. Students tended to identify and sort shapes on an "it looks like" basis. Initial descriptions of shape were generally imprecise and level 0 in nature, even for students who later exhibited level 1 thinking. It was only after some instruction that students began to express themselves more precisely in terms of properties of shapes. There are several possible explanations for this level 0 thinking. One is the lack of experience in doing geometry in school. A second is that even when geometry was studied, it was probably done at level 0. The analysis of three major USA text series in this study indicated that most geometry material in grades 1-6 was at level 0. Pyshkalo (1968) reported similar findings in analyzing traditional Soviet curriculum. A third explanation is that the entry assessment tasks which involved cut-out figures or diagrams and can be responded to at levels 0, 1 or 2, are done most naturally at level 0 which matches the format in which they were presented. Also, since the interviewer accepted level 0 responses as correct and reinforced them with comments such as "okay....good..", the student may have been led to believe that this was the expected kind of response.

The the entry level of the sixth graders was mainly level 0 or at

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ines level 1, potential level for all but three of the 16 sixth raders was at level 1 or even level 2. This lack of progress into evel 1 maybe attributed to: low general ability; difficulty with anguage, such as the use of quantifiers and conditional xpressions; and the absence of many prerequisite geometric and easurement concepts. The instructional materials in Module 1 were esigned to review topics normally covered in grades 4-6, not evelop them for students with weak backgrounds. Also, the nterview schedule did not permit time needed to carefully develop opics with these 3 students.

esults involving the sixth graders who made progress into level 1 nd 2 are noteworthy for several reasons. First, since these tudents performed at a higher level on potential level assessment asks than on entry tasks, conventional assessments of level may not dequately characterize a student's ability to think at certain evels. Second, it should be noted that the interviewer and instructional materials seemed to play a critical role in helping tudents to progress within a level or to a higher level. Also the nterviewer at times guided student responses through questioning nd directives about the quality of responses, helping students to learn the "rules of the game." Students on a level realize this, but students in transition need guidance about expectations.

Finally, these results suggest that some sixth graders are quite capable of engaging in geometry activities that call for level 1 and even level 2 thinking. Fyshkalo (1968) reported similar findings. As noted above, unfortunately USA textbook material on geometry is

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aly at level 0, providing students with little opportunity to
progress in level of thinking. Thus students may not be
pared adequately for secondary school geometry which involves
el 3 thinking. This supports van Hiele's contention that
ficulties in tenth year geometry are related to a gap in level of
nking created by insufficient experiences at levels 0-2. Closing
gap by improving the geometry curriculum and instruction,
ticularly in grades 4-8, is a challenge that the Soviets
cators have already acknowledged and one that USA mathematics
cators are beginning to accept, in particular "levelists" who
d the Van Hiele model a useful tool for addressing issues about
rning and thinking in geometry.

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THE ATLANTA - EMORY LOGO PROJECT: TEACHING AND UNDERSTANDING GEOMETRIC RELATIONSHIPS THROUGH LOGO

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This project is a two year investigation into the use of LOGO for helping ninth grade students develop understanding of geometric relationships. The van Hiele model of geometric thought suggests that students' lack of understanding of geometric relationships may account for much of the failure in high school geometry courses. The LOGO computer language, with its use of Turtle graphics, has the potential for helping students develop this understanding. LOGO learning environments, based on a theory of relational learning cycles which emerged from a synthesis of the SOLO taxonomy and Skemp's model of mathematical understanding, will be created for groups of ninth grade students and their teachers in three Atlanta schools. The investigators will determine whether the intervention is successful in improving students' geometric understanding. The research is supported in part by an equipment grant from the Apple Education Foundation.

RATIONALE

Recent national assessments of the mathematical ability of American high school students (Carpenter et al., 1983) have highlighted a serious problem in the development of mathematical thinking: Students are failing in situations which require mathematical reasoning skills and understanding beyond the level of basic skills and memorization.

The problem is particularly critical in the area of geometric understanding. Formal high school geometry appears to be a critical turning point for many students in their mathematics studies. Less than one-third of American high school students take mathematics courses beyond tenth grade geometry (NASSP, 1983). Formal geometry demands abstract reasoning abilities and an understanding of geometric relationships.

Part of the problem is the way geometry is both taught and learned in our public schools. Since the curriculum development era of the 1960's, geometry

has been introduced into the school mathematics curriculum in the elementary and middle school grades. However, geometry instruction in the early grades usually consists of identifying shapes and stating properties of geometric figures at a recall level without attending to more complex concepts that require understanding of relationships among geometric figures and their properties. The National Assessment of Educational Progress (NAEP) indicated that:

> Although most students at all age levels could identify common geometric shapes, relatively few demonstrated a knowledge of basic properties of these shapes. ... Fewer than 20 percent of the 17-year-olds could calculate the area of either a triangle or a parallelogram. (Carpenter et al., 1983, p. 655)

VAN HIELE LEVELS

The van Hiele theory of thought development in geometry asserts that there exist five hierarchical levels of thinking that a successful learner in geometry must achieve: 1) shapes, 2) properties, 3) relationships, 4) deduction, and 5) abstraction. The following is a description of the hierarchy:

> The first level is one of recognition of geometric figures as entities, without any awareness of parts of figures or relationships between components of the figure. In the second level, the components of a figure can be discerned and the properties of a figure described but not formally defined. Also a student at this level may recognize that two figures have properties in common but s/he does not conclude, for instance, that a rectangle is also a parallelogram.
>
> In the third level, relationships among the properties of a figure are established as are relationships among the figures themselves. The possibility of one property following from another is known, and a logical partial ordering of classes of figures occurs. It is only in the fourth level, however, that the logical structures of analysis and of proof are grasped and that deduction is understood as a means of constructing a geometric theory. The fifth level is characterized by standards of rigor and abstraction represented by modern geometries. A student at this level develops a geometric theory without reference to concrete applications. (Skypek, 1982, pp. 1-2)

Van Hiele (1958) believes that these levels are achieved largely as a result of instruction and that students who have not had sufficient, appropriate instruction in geometric relationships, for example, cannot be successful in

formal geometry courses which require proofs and deduction.

SOLO/SKEMP SYNTHESIS

In addition to the problem of curriculum sequence and content in geometry instruction, there exists a problem in the dynamics of the learning environment. As indicated by the results of the national assessment (Carpenter et al., 1983), students are not learning the reasoning skills and conceptual understanding necessary for effective application of mathematical ideas. This problem of the quality of the learning environment will be addressed in this study by strategies for curriculum development, teaching methodolgy and assessment based on a synthesis of the SOLO taxonomy with Skemp's model of mathematical understanding, and by use of the LOGO computer language to design a geometric microworld which embodies the necessary dynamics suggested by this synthesis.

The SOLO (Structure of Observed Learning Outcomes) Taxonomy (Biggs & Collis, 1982) consists of five levels described as follows:

1. Prestructural: the learner does not engage the task or gives completely unassociated data.

2. Unistructural: use of one piece of information only in responding to the task (formation of a single datum).

3. Multistructural: use of several unrelated pieces of information (acquisition of parallel data).

4. Relational: integration of the separate pieces of information to produce a viable solution to the task.

5. Extended Abstract: derivation of a general principle from the integrated data which can be applied to new situations (acquisition of a higher-order concept).

Biggs and Collis's own integration of the SOLO levels into Piagetian developmental stage theory produces a model of learning cycles similar in many ways to the van Hiele model of geometric thought development and to Skemp's model of mathematical understanding.

Skemp (1976) proposed the existence of two types of mathematical understanding which could be generated by mathematics learning and teaching in schools:

instrumental and relational understanding. Instrumental understanding is the product of rote learning of rules and theorems, and their specific applications. Relational understanding is the product of a learner's personal involvement with mathematical objects, situations, problems and ideas.

A synthesis of the SOLO and Skemp theories (Olive, 1983) suggests that learners acquire new understanding (of a subject) by going through a learning cycle based on the SOLO Taxonomy in one of two ways: instrumentally or relationally. At each stage in a relational learning cycle (unistructural through extended abstract) the learner is personally involved with the available data. The data are, in fact, products of the learner's own investigations. In contrast, the data available in an instrumental learning cycle are given to the learner to memorize by some external source of information (usually the teacher, textbook or computer).

At the end of a relational learning cycle the learner enters a higher mode of functioning. The newly acquired higher-order concept becomes the single datum on which to build the next learning cycle in the higher modality. The extended abstract level is indicative of this modality shift and, according to Skemp (1979), can only be achieved by reflective activity - the learner needs to think about his/her thinking: What do I know? How does it fit together? If I change this, how will it affect that? Will it work in different situations? How did I get here from there? What if...? etc. Being able to relate personally to a learning experience, being able to relate the experience to existing knowledge (and previous experiences), and reflection by the learner on what he/she knows are the key ingredients of a relational learning cycle.

By applying this model of qualitatively different learning cycles to students' experiences in school geometry, a rationale for the currently high rate of failure can be generated. Without experiences at the (van Hiele) level of "geometric relationships" the higher-order axioms and theorems of formal geometry can only be learned instrumentally.

The goal of this study is to design an intervention which will help to bridge the existing curriculum gap and help to break the repetition of instrumental

learning cycles by providing a relational learning environment based on the LOGO computer language and Turtle geometry.

A LOGO LEARNING ENVIRONMENT

Papert (1980), director of the MIT LOGO project, sets forth a thesis for the learning of school mathematics that is similar to those of van Hiele and Skemp. He contends that "What an individual can learn, and how he learns it, depends on what models he has available" (p. vii). Papert later states:

> Our educational culture gives mathematics learners scarce resources for making sense of what they are learning. As a result our children are forced to follow the very worst model for learning mathematics. This is the model of rote learning, where material is treated as meaningless; it is a dissociated model. (p. 47).

Papert and his colleagues created a new area of mathematics which they termed "Turtle geometry," as part of the LOGO computer language. Papert claims that Turtle geometry is "a better, more meaningful first area of formal mathematics for children [than traditional geometry]" (p. 51).

The Turtle exists both as a physical robot and as a computer graphic simulation of the physical embodiment. It is a geometric object having the properties of both position and heading. These geometric properties (which constitute the mathematical state of the Turtle) can be changed by simple commands issued to the computer via the keyboard: FORWARD, BACK, RIGHT, LEFT. Each command is accompanied by a numeral which indicates the amount of change (the number of Turtle steps for distance, the number of angular degrees for turning). Thus the commands: REPEAT 4 [FORWARD 50 RIGHT 90] would cause the Turtle to draw a square of size 50 Turtle steps.

Exploring Turtle geometry gives the student an opportunity to relate personally to angle measure, relationships between different geometric shapes and their properties, and all geometric transformations. Because of its concreteness, Turtle geometry allows students to relate these mathematical ideas to their own physical experiences. The expandability and structure of the LOGO language allow students to create building blocks for solving complex

problems. This "building block" property of the language encourages progression through SOLO learning cycles. The activity of "teaching the Turtle" encourages students to think about their own learning process. This reflective processing is necessary for the construction of the higher-order concepts involved in formal geometry. Thus, LOGO (by its nature) has the potential to generate relational learning cycles; however, an analysis of previous LOGO projects (Watt, 1982) indicates that it has not done so for some students in previous studies. By structuring the learning environment and basing the teaching sequence and methodology on the SOLO/Skemp synthesis, this research project intends to tap the potential of LOGO for generating relational learning cycles.

RESEARCH DESIGN

LOGO learning environments, based on Turtle geometry, will be created in three inner-city schools, for approximately 150 ninth grade students and their teachers. Students and teachers will explore geometric relationships through a series of Turtle geometry projects for a period of 18 weeks. Control groups of ninth grade students, who will not be exposed to LOGO, will be identified in each school.

All student interactions with LOGO will be captured on disks using dribble files. These data files will be analyzed using the SOLO taxonomy in order to determine each student's progression through SOLO learning cycles. Students' entering and exiting van Hiele levels will also be evaluated.

During the second year of the project, comparisons will be made between the LOGO students and the control groups in terms of their performance in and attitude towards tenth grade geometry. Drop-out rates will also be compared.

The results of this study should have important implications for understanding student thought processes, and for determining the usefulness of the LOGO environment in developing geometric understanding. Such implications could, in turn, impact on computer instruction and geometry curricula in middle and high schools in the U.S.A.

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THE GROWTH OF SIMILARITY CONCEPTS AT THE SIXTH GRADE LEVEL

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Many children never reach advanced stages of cognitive development in proportional reasoning. Studying similar geometric shapes would seem to provide children with more concrete mental images for other types of proportional reasoning and have important pedagogical implications. The purpose of this pilot study is to determine the extent to which sixth graders exhibit an understanding of concepts of similarity before and after a three-week unit of instruction. Two classes were taught a unit on similarity developed by the Middle Grades Mathematics Project. Prior to and after instruction students were given two paper-and-pencil tests on concepts of similarity and on general proportional reasoning. From results of these tests, students were selected for pre-post interviews. Interviews indicate individual strategies and cognitive levels of thinking about similarity as a result of instruction. The most impressive finding is the lack of consistent strategies within individuals.

INTRODUCTION

The concept of similarity is important as an essential stage in the development of children's geometrical understanding of their environment, and as a concrete aspect of proportional reasoning. Phenomena that require familiarity with enlargement, scale factor, projection, area growth, indirect measurement are frequently encountered by children in their environment. Because of its visual representation, similarity may be a first step towards an understanding of proportional reasoning. Inhelder and Piaget (1958), consider proportional reasoning as one of the six abilities that characterize the formal-operational thinker. Although proportional reasoning is widely required in everyday life, the majority of children lack the ability to use such reasoning effectively (Karplus & Karplus, 1972; Pagni, 1983).

This study is concerned with finding patterns in children's perception of similarity and will attempt to indicate effects of instructional intervention, i.e., use of the Middle Grades Mathematics Project (MGMP) Similarity Unit at the sixth grade level.

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METHODOLOGY

Two classes of sixth graders with the same mathematics teacher comprise the sample for the study. The classes were homogeneously grouped with the higher ability students in one and the average ability in the other. In-depth interviews were conducted with 17 students who scored in the middle range on two paper and pencil tests given pre- and post-instruction. These tests were the NCMP Similarity Test and a Ratio and Proportion Test which included the Karplus Mr. Tall/Mr. Short problem and selected items from the Concepts in Secondary Mathematics and Science (CSMP) Ratio and Proportion Test. Each audiotaped interview lasted from 30 to 60 minutes. The tasks to be performed were presented to each student in a uniform way. The interviewer asked students to explain their reasoning in detail. Students were presented tasks focusing on similarity of rectangles and those focusing on area growth. The two sets of results will be discussed separately.

The first part of the interview presented students with four different kinds of tasks related to determining similarity of rectangles. Each of these tasks was varied along a numerical scale designed to test the student's facility with handling proportions of increasing numerical difficulty. Table 1 shows the four tasks and the four ways in which the numbers in the proportions were varied.

Table 1. Interview questions

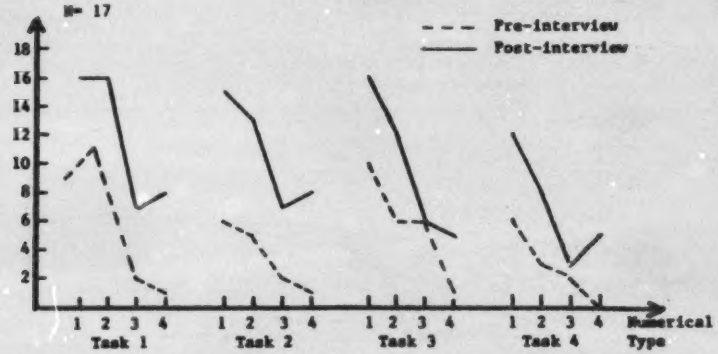
TASK \ NUMERICAL TYPE (a by b / c by d)	1 a | b a | c	2 a ≠ b a | c	3 a | b a ≠ c	4 a ≠ b a ≠ c
1. Decide whether two drawn rectangles are similar or not.	3 by 6 and 9 by 18	2 by 3 and 8 by 12	3 by 9 and 4 by 12	6 by 8 and 9 by 12
2. Decide whether two cut-out rectangles are similar or not.	2 by 4 and 6 by 12	2 by 3 and 6 by 9	2 by 6 and 3 by 9	4 by 10 and 6 by 15
3. Given the lengths of three sides of two similar rectangles, find the fourth side.	2 by 6 and 6 by ?	2 by 5 and 6 by ?	4 by 12 and 7 by ?	6 by 10 and 9 by ?
4. Cut a strip to make a rectangle similar to a given one.	2 by 4 and 6 by ?	2 by 3 and 8 by ?	2 by 6 and 5 by ?	4 by 6 and 6 by ?

econd part of the interview dealt with the concept of area growth in ar figures. Students were shown a rectangle representing a small room cost $300 to carpet. The interviewer asked the student what would e price of carpeting a larger room which is twice as long and twice ide. The task was presented on three levels of concreteness: (1) no ional illustration, (2) student is given an assortment of shapes from to choose a correct representation of the larger room and (3) student ven several copies of the small room and asked to represent the larger The question was repeated for a room enlarged by a scale factor of

MAIN RESULTS

ts related to the four tasks that involved the concept of similarity een rectangles will be described first, and performance on the task ving the area relationship of two similar shapes will be described next. o the small number of subjects, the indicated trends should be considered caution.

arity of Rectangles. Student performance on the interview items related ectangle similarity varied widely as a function of both task and the of numbers involved. Figure 1 illustrates the level of success on item expressed by the number of answers given that employ proportional ning. A general improvement in performance as a result of instruction ticeable.



re 1. Level of success for rectangle similarity items.

With some irregularities the level of success of the two groups tends to decrease as numerical difficulty increases. By collapsing results across tasks, a post-instructional level of success of 87, 72, 34 and 38 percent was observed on items involving proportions of numerical types 1, 2, 3, and 4 respectively.

Thus, at sixth grade level, more than two-thirds of the students used proportional reasoning when the external ratios (i.e., width to width, and length to length) was an integer, but fewer were able to do so in other cases (even if internal ratios of length to width were an integer). Similar differences in performance as a function of the numbers involved in other proportionality tasks were indicated by Abramovitz (1975), Karplus et al. (1980), and Quintero (1983).

Level of success in student performance varied as a function of task. Ordering of tasks according to level of success was difficult especially for pre-instructional performance. The results after instruction on similarity indicate a level of success of 69, 63, 57, and 41 percent on tasks 1, 2, 3, and 4 respectively.

These results seem to indicate that Task 1 (deciding similarity for pairs of drawn rectangles) was the least difficult, and Task 4 (creating a rectangle similar to a given one by drawing its fourth side) was the most difficult. Analysis of interviews allowed for the following categorization of student reactions to the presented similarity tasks:

* Proportional Reasoning--setting up two ratios and a correct or incorrect consideration of their equivalence, or more frequently, considering the scale factor by which the small rectangle is enlarged.

* Whole Multiplication--"fitting in" the sides of the small figure a whole (but not necessarily the same) number of times into the sides of the bigger figure. This kind of reasoning leads characteristically to the conclusion that if the scale factor is not an integer, the figures are not similar.

* Multiplication and Adjustment--multiplying to enlarge, and "adjusting" by subtraction or addition (e.g., in the proportion $2:6::5:x$, $x = 13$ because $5 = 2 \cdot 2 + 1$, and thus $2 \cdot 6 + 1 = 13$).

* Addition--considering the difference rather than the ratio of the numbers.

* Visualization--using intuition without considering the lengths of the sides.

With the exception of whole multiplication, these categories correspond to the classification of strategies used in other proportional reasoning tasks reported by Karplus and Karplus (1972).

The deterioration of performance according to increasing difficulty of numerical type of proportion indicates a remarkable lack of within-subject consistency in the employed strategies. Frequently, during four consecutive interview questions, students regressed within the same task from proportional reasoning to whole multiplication or multiplication/addition, and finally to using the additive strategy as the numerical type of the proportion became more difficult. Lack of consistency may be observed from the fact that no students used the same strategy for all sixteen items presented. During the pre-instructional interviews, one subject used four different strategies for the same task presented consecutively in four different numerical instances; eight subjects used three strategies for at least one task; seven subjects used two strategies for the same task; and only one student was consistent within each task. Instruction improved to a certain degree the within-task consistency: eight subjects used three different strategies for at least one task, seven subjects used two, and two subjects used proportional reasoning throughout the interview.

Area Growth. Tasks related to the area relationship of similar shapes had as a purpose the detection of student thinking strategies. Tasks were administered to the eight students in one of the two classes. Instruction did not improve performance on this task. The following thinking strategies could be detected:

* Area Scaling--increasing the area by counting the number of times the small rectangle "fits into" the larger one. No student used the square of the linear scale as an argument.

* Scaling--increasing the area by a factor which is different from the area, or the linear scale.

* Scaling and Adjustment--the scaling is followed by an additive "adjustment".

* Periferal Counting-- considering the area of the original rectangles that "fit in across and down" the increased rectangle (characteristically counting the corner-rectangle twice) and ignoring the interior of the enlarged shape.

* No Reasoning--not being able to explain, or not giving an answer.

Lack of consistency in employed strategies was noticeable in this case as well. Strategies tended to improve as the level of concreteness increased from an unillustrated presentation to the students' building of the enlarged rectangle with a number of cut-out original rectangles. It should be noted that at the post-instructional stage, fewer than 50 percent of the students were successful in this task even with this last presentation mode. The next phase of the study will be to expand the research to grades seven and eight, and to more schools and students.

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PROBLEM SOLVING

THE IMPORTANCE OF BELIEFS AND EXPECTATIONS IN THE PROBLEM SOLVING PERFORMANCE OF SECOND GRADE PUPILS.

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It is argued that a problem solver's beliefs and motivations are intimately related and that they influence both what can count as a problem and as an acceptable solution. Other influences include persistence, willingness to take the initiative, means of judging competence, and means of achieving satisfaction. The possible relationship between beliefs and heuristics is discussed briefly. Analogies drawn from the philosophy and sociology of science suggest that the analysis might have some generality.

In recent years, several theorists have argued that children's problem solving behavior is influenced by their general beliefs about the nature of mathematics (e.g. Confrey, 1984; Schoenfeld, 1983; Silver 1982). There is also some suggestion that these beliefs might be intimately related to children's general motivations for doing mathematics (Cobb, in press a,b). My purpose is to explore the role that beliefs and motivations played in the problem solving activity of two second grade children. Wheatley has conducted a similar analysis with sixth grade children in a companion paper.

BELIEF AND ANTICIPATION

There is a consensus among contemporary philosophers of science that the scientists actions are constrained by a largely implicit contextual framework. These frameworks, which are an expression of largely implicit beliefs, guide anticipations as scientists construct, elaborate, and attempt to solve problems. In essence, problem solving involves making one's anticipations work and these anticipations are constrained by one's beliefs. Knorr (1980), for example, concluded that the "bright ideas" that the precipitated investigations conducted

by a group of biochemists were potential or unrealized solutions rather than hypothesis or ex ante conjectures.

Beliefs seemed to play a similar role in the mathematical problem solving behavior of two second grade children, Scenetra and Tyrone. This became increasingly apparent when the children, to whom similar concepts of addition and subtraction were attributed, sometimes behaved in radically different ways while solving specific tasks. Their differing beliefs constrained both what could count as a problem and what could count as an acceptable method of solution.

PROBLEM SOLVING METHODS

In one session, both children engaged in an activity in which they transferred marbles from one of two cups to the other. Scenetra was soon able to say how many marbles were in each cup and how many there were in all without counting. She could also generate the next of a sequence of compensating addends in certain highly specific situations. However, the lack of generality and flexibility of solutions indicated that they were not an expression of underlying numerical relationships. Instead, she focused on and abstracted regularities from the number words involved in successive tasks.

Tyrone appeared to be less competent than Scenetra when he attempted to solve these tasks. However his gestures indicated that he continued to represent the transfer of a marble, suggesting that he strove to elaborate a structure of numerical relationships. Tyrone did not seem to be aware of or consciously rejected superficial number word regularities. Instead, his belief that a solution should express conceptual structure was implicit. Following Skemp (1976), Tyrone's beliefs about mathematics are said to be relational. Scenetra's solutions to these and other tasks indicate that the sole criterion by which she usually judged a method was whether or not it yielded a correct answer (i.e. she received appropriate feedback from the teacher). Her beliefs are said to be instrumental -- methods were valued only as instruments to achieve correct

answers.

WHAT CAN COUNT AS A PROBLEM

Tyrone solved the sentences 44-11=__, 45-12=__, and 46-13=__ by counting and found that they had the same answer, 33. He then turned to the teacher and asked, "Why?" This number word regularity was problematic because it was not, for him, an expression of underlying conceptual structure; it was something that had to be explained. As we have seen, the identification of number word regularity did not give rise to a new problem for Scenetra. Instead, it terminated her problem solving activity. The crucial differences between Scenetra's and Tyrone's problem solving behavior is captured by Silver's (1982) contention that

> a person who believes that there is an underlying structure to mathematics and that this structure is more important than surface details will approach the study of mathematical material quite differently than a student who does not hold this belief. (p.21)

HEURISTICS AND SUBCONTEXTS

Minsky (1975) contended that

> ... any problem a person can solve at all is worked out at each moment in a small context and that the key operations in problem solving are concerned with finding or constructing these working environments. (p.119)

By using a heuristic, the solver delineates a subcontext within which to elaborate and explore anticipations. A heuristic embodies anticipations about the potential fruitfulness of operating in a particular subcontext and guides the solver as he or she strives to construct an unrealized solution. Wimsatt (1981) noted that many of the heuristics that guide the scientist are implicit metacognitive prompts (cf. Silver, 1982) rather than consciously formulated principles. Both children seemed to use various implicit, elementary heuristics. Tyrone, for example, frequently searched for opportunities to relate a current task to a known sum or difference. Scenetra, in contrast, frequently focused on syntactical features of a problem statement or of a sequence of answers.

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A further difference between the children concerned flexibility. Tyrone frequently attempted to use a variety of methods (e.g. counting, thinking strategies) when he experienced difficulty. The possibility of operating in an alternative subcontext rarely occured to Scenetra unless prompts were given.

MOTIVATION

The two children seemed to have very different motives for engaging in mathematical activity. Scenetra's quest for correct answers seemed to be related to her desire to look smart or to avoid looking stupid. Nicholls (1983) called this form of motivation ego-involvement. "The child is preoccupied with herself ... rather than with learning, understanding, or finding out ... learning, as such, is not valued. Learning is not an end in itself". (p.213) While she strove to get correct answers, understanding per se was not her goal. Instead, it was to anticipate and thus control how others would act by producing what she believed to be appropriate behavior. Mathematical problems were subsumed by problems relating to the interpersonal dynamics of the classroom situation. Scenetra's problem solving activity also evidenced ego-involved children's tendency to judge their competence by comparing their performance with that of their peers. Failure did not give rise to new problems but was interpreted personally and led to self-doubts about her competence. Not surprisingly, she readily gave up when things did not work out fairly quickly. Persistence entails the risk of further failure, and Scenetra was not prepared to gamble with her self-esteem in this way.

Tyrone, in contrast, was motivated by a desire to understand and to gain mathematical insights. The activity of solving mathematical problems was an end in itself. Nicholls suggested that children with this form of motivation, called task-involvement, judge their competence relative to their previous levels of performance or understanding. Satisfaction comes from the achievement of relational understanding rather than from demonstrations of superior ability. As initial failures led to a new, more demanding intellectual challenge for Tyrone, he was far more persistent than Scenetra. He was also prepared to take

the intiative, often suggesting problems that provided him with a suitable challenge. He also depended less on the teacher for directions and guidance and, on several occasions, told the teacher, "You don't help me no more."

CONCLUSIONS

Observations reported in this and other papers suggest that the orig[illegible] of instrumental beliefs and concomitant motivations can be traced to the early grades. Confrey (1984) suggested that the public nature of both formal and informal evaluation methods, the pace at which the teacher covers the text, and the teachers' role as an external authority all foster these beliefs and motivations. While the processes by which children both reorganize and sustain their belief systems are beyond the scope of this paper, it seems clear that these are as much a sociological as a psychological phenomenon (cf. Confrey, 1984).

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The Importance of Beliefs and Expectations in the Problem Solving Performance of Sixth Grade Pupils.

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Abstract

Based on the assumption that students construct knowledge for themselves, the beliefs, expectations, and confidence of grade six pupils are examined. An analysis of protocols from two nonroutine problems served as a basis for understanding factors in problem solving performance. Differences in the responses of students with and without heuristic problem solving experience were apparent. Implications for classroom practice are discussed.

Introduction

It is likely that beliefs and expectations play a major role in the problem solving performance of students. (Cobb, in press; Confrey, 1984; Piaget, 1964; Silver, 1982; Skemp, 1979). Confrey (1984) states that, "It is our belief that success in promoting the development of problem-solving strategies must be accompanied by changes in students' expectations of mathematics and mathematics learning." p.2. If students view mathematics as a set of prescribed procedures they will be perplexed and confused by tasks for which set methods will not suffice. If on the other hand they believe mathematics is comprehensible and relational, they are more likely to respond with interest and confidence to problem situations.

In a study by Wheatley and Wheatley (1982), 102 sixth graders were observed as they solved nonroutine mathematics problems. Sixty-eight of these subjects had experienced five months of heuristic problem solving while the other 34 had continued to study mathematics in conventional ways. An analysis of the student interviews revealed that the control subjects were quite unsuccessful in problem solving; their average problem solving score on a five problem test was 4.4 compared to 14.3 for the experimental subjects. During the interviews it became clear that the subjects who had not studied heuristic problem solving expected to apply a rule. If they could not identify a rule to use they would either just write down a number, knowing it was wrong or say they did not know what to do. For example, none

of the 34 control group subjects was able to solve the problem shown below.

Problem one: The sum of two numbers is 33. Their difference is 15. What are the two numbers?

The fact that none of the 102 subjects in this study were able to solve this problem by operating directly on the numbers, even if successful, makes this a nonroutine problem for them. Grade six pupils who had not been exposed to problem solving heuristics would typically subtract the two numbers without any plan for using the result. Once they had performed the subtraction, 33 - 15 = 18, they did not know what to do next. They seemed puzzled and often wrote 18 as the answer. Others added 33 and 15 and wrote 48 as the answer. The degree to which students used heuristics seemed to be a function of their beliefs and expectations. The control group subjects' past experience led them to think they could succeed by performing an operation on the given numbers in the statement of the problem. When this expectation was not satisfied they were unable to proceed. They did not elaborate a problem representation.

In contrast to the control group subjects, many of the students who had studied heuristics approached the problem in quite a different manner. While they would frequently make the same initial move, 33 - 15 = 18, their response to this move was of another character. They did not act as though they expected the result to be the answer. To them, the subtraction was viewed as but one exploratory move in the process of understanding the problem. That is, they had different expectations about their work. They followed the subtraction with other computations. On the problem shown above, the experimental group made an average of 13 computations compared to 2 computations by the control group. The experimental subjects made the key move of introducing numbers not given in the statement of the problem. They solved the problem by trying pairs of numbers until both conditions were met. There was evidence that students were attempting to elaborate a problem representation to understand the problem.

The following protocol typifies solution methods observed among the experimental subjects.

he sum is 33."

```
  33
- 15
----
  18
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That doesn't help."

Maybe the numbers are 15 and 18. No, can't be since that ould be a difference of only 3. The numbers must be urther apart. "

I'll try 20 and 13, they add to 33."

The difference is 7 - that's closer."

The numbers must be closer together, no, further apart."

How about 22 and 11. The difference is 11."

Try 24 and 9. Hey, That works!"

n this solution, the individual is elaborating a problem representation as he solution progresses. It is clear that this student expected to find a pattern. He believed that mathematics is orderly and predictable. Furthermore, he realizes that he can construct a solution path for himself rather than identify the "correct" way to do the problem.

Expectations

This contrasting behavior could be explained in terms of different expectations. The heuristic problem solvers expected to explore the problem, to build a mental representation of the problem while the students without heuristic experience expected to operate on the given numbers and arrive at the answer. Furthermore, the experimental subjects seemed to approach the tasks relationally while the control subjects responded instrumentally (Skemp, 1979).

The contrast is also apparent in the following problem.

Problem two: The distance between the tips of my fingers is 65 inches (problem is demonstrated with outstretched arms). Each arm is 24 inches long. How far is it across my chest?

One might have expected this task to be unproblematic for children at these grade levels. However, fifth and sixth grade students without heuristic experience typically write as their answer, in a short period of time, either 41 inches or 89 inches; they subtract or add the two numbers in the problem

and write down the answer without any further thought. Evidently, conventional mathematics instruction builds a set of expectations that the answer to a word problem is obtained by first determining the operation (addition, subtraction, multiplication, or division) and then performing that operation on given numbers at which time the answer will be reached. Analyzing the task just did not occur to them. On problem one, Ignacio asked, "Do I add?" He did not expect to make exploratory moves in arriving at an answer. He did not expect to introduce other numbers into the problem (where would they come from?). He did not expect to test the number obtained as an answer. He did not expect to devise a method of approaching and solving a problem. When he was stuck, the only thing he knew to do was ask the teacher for help. If that was judged inappropriate, he might just write a number to have "finished" the problem.

On the other hand, students who had experienced heuristic problem solving made a variety of computations which indicated that they had constructed a representation and were testing potential solutions. The contrast in behavior with students in the experimental group was striking. Less than 10% of the control subjects made exploratory moves while 68% of the students with heuristic problem solving experience showed evidence of thinking through the problem and building a representation. Their expectations were clearly different.

Confidence

In the interviews, the confidence level of the subjects was explored. After each problem had been completed, the subject was asked to indicated on a scale of one to five how confident he or she was of the answer. An interesting pattern of responses emerged. Students who had no idea how to approach the problem and just wrote an answer without any supporting reasoning, would say they were very confident of their answer. We interpreted this response as a desire to be finished with the problem because they had no idea how to solve it and anything but an expression of total confidence might invite further inquiries and thus reveal their perceived inability to cope with the task.

One student in the experimental group that was observed carefully over a five month period was initially very insecure in problem solving. She did not want us to see her work and she was reluctant to respond to our questions. She seemed concerned about appearing foolish or stupid. This

ego orientation (Nicholls, 1983) influenced her work in dramatic ways. "Once she realized that she was not expected to solve the problem in a prescribed way, she relaxed and tackled the problems." (Wheatley and Wheatley, 1982, p. 83). She became less answer-oriented and more confident. This pattern of behavior was observed in numerous other students.

It is likely that problem one proved difficult to grade six students because they had studied from a rule oriented curriculum. In problem solving one does not usually proceed directly according to a fixed procedure in arriving at a derived result. At times it is productive to construct a list of numbers from which the answer can be determined by inspection. School mathematics experiences do not prepare children for such reasoning and this is precisely the limitation of rule oriented teaching. We can develop students that are good problem solvers but the process must include attention to beliefs and expectations.

Implications

A major goal of educators and parents alike is reasoning ability; all want students to be good problem solvers. It is also widely accepted that computational skills are only useful if students can apply their skills to problems. Yet the educational methods currently exemplified by conventional textbooks and teacher instruction is not effective in developing mathematical reasoning. It appears that students form a set of beliefs about mathematics and expectations which distract them from focusing on the problem itself. By focusing on the beliefs and expectations of students, it is possible to gain a deeper understanding of the reasons for their performance. Classroom practices which deemphasize competition, teacher evaluation, and provide the opportunity for students to interact are likely to be useful in helping students construct useful knowledge frameworks. Furthermore, by designing instruction to build a view of mathematics as a network of comprehensible and related ideas, we can help students become successful students of mathematics.

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MATHEMATICAL PROBLEM-SOLVING PROCESSES OF AVERAGE-ABILITY MIDDLE-GRADE STUDENTS WORKING IN SMALL GROUPS

by

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Abstract

The purpose of this clinical study was to describe the processes used by middle-grade average-ability students solving applied mathematical problems. Working in groups of three, subjects were video-taped solving a problem. Audio-tapes were made of groups discussing their sessions. Results suggest that average-ability students exhibit "raw" heuristics that are not helpful when the student is unable to form a useful representation of the problem. Factors that inhibited the process were identified. The group proved to be a useful vehicle for problem-solving research and may prove to be a useful pedagogical format.

Purpose

The purpose of this clinical study was to describe the processes used by average-ability middle-grade students working in small groups solving applied mathematical problems.

Rationale

The rationale for choosing the 'average' student in this investigation was in response to the relative dearth of such a focus in problem-solving research. Lesh (1981) states that traditional problem-solving research has attempted to identify and analyze the processes of bright students with the goal of improving instruction for the average student. He goes on to say,

> The qualitatively different systems of thought used by gifted problem solvers may be just as inacessible to normal problem solvers as formal operational reasoning is to most seven-year-olds. (p. 239)

The rationale for selecting an applied problem was a result of two factors. First, recent results of the National Assessment of Educational Progress in Mathematics (Lindquist, et al., 1983) found "no gain on exercises assessing deep understanding or application of mathematics" (p. 14). Students are still unable to apply skills in mathematics. Second, frequently in school students are taught a concept then asked to apply it. In the world they are faced with a situation and must try to find a mathematical model to fit it. If students are to improve in their ability to apply the mathematics they have learned, we will need to provide experiences that more closely mirror the real world situations they will be faced with.

The rationale for using a group as the data collection unit is also in response to two factors. Silver (in press) points out

that the "small group may provide an opportunity to study the externalized internal dialogue that a problem solver might engage in." Secondly, Noddings (1983) has had success in using the group as a vehicle to improve problem-solving success. She found that students working in small groups learn problem-solving strategies from their peers.

Methods and Procedures

Twelve average-ability seventh-grade students from a middle-class urban elementary school were subjects. Four groups, consisting of three students each, met three times. At the first meeting individuals were given two instruments, a Problem Solving Sort Task and a Problem Solving Attitude Questionnaire. At the second meeting the groups were video-taped solving an applied mathematical problem. At the third meeting a follow-up discussion with each group was audio-taped.

Percentages were calculated for reporting students' responses from the Problem Solving Sort Task and the Problem Solving Attitude Questionnaire. The video-taped and audio-taped sessions were transcribed and analyzed to describe the processes and to develop categories of factors that influenced the problem-solving process. As a category emerged in the transcripts, a hypothesis was formed. Verification was sought not only in the transcript of the problem-solving session but also in the other three data collection sources.

Results

The average-ability students in this study exhibited "raw" heuristics without prior instruction (Silver, in press), for example numerous instances of draw a diagram, reread the problem, think of a specific case, restate the problem, and trial and error were observed. However, these heuristics were generally not helpful to the student who was unable to form a useful representation of the problem. Since the goal of the study was to describe the processes used by average-ability students solving problems, merely listing the unsuccessful heuristics was not sufficient. In order to develop a clearer picture it was necessary to isolate categories of factors that hindered the problem-solving process. The following were identified: (1) lack of experiential background; (2) imposing unrequired restrictions on the problem; (3) lack of understanding of both mathematical and situational concepts; (4) lack of metacognitive processes; and (5) influence of belief systems.

Working in a group offset some of these factors. The group frequently supplied background information that individuals did not possess. The disbelief of peers acted as a form of monitoring when self-monitoring was not present. Group members challenged comments that diverted from the restrictions of the problem. Finally, the group allowed time for students to think and consider answers and suggestions.

The beliefs that this group of students brought to the problem-solving sessions were overt in their responses to the Problem Solving Sort Task, the Problem Solving Attitude Questionnaire and the audio-taped sessions. For example, the subjects felt that "word problems" were harder to do and were therefore less liked. This was said directly in the interviews and was supported by the two instruments. However, other beliefs were more obscure. The student who rejected a solution that used only some of the numbers in the problem and countered with a solution that used all the numbers may have been espousing a belief that "in word problems you must use all the numbers."

Hypotheses

The conclusions drawn from this research can only be made about the subjects who participated in the study. However, hypotheses can be developed from these conclusions. The following discussion is such an attempt.

Experiential background within and outside of the classroom affects not only the problem-solving ability but the belief systems that influence problem-solving ability. Students who enter the problem-solving session without sufficient experiences to create a useful representation of the problem may; (1) not have the cognitive structures necessary to build the new structures needed for the solution path, or (2) may simply have unstable cognitive structures. Also, a student's environment has

fostered beliefs, either overtly or covertly, that influence some of the decisions that are made in a problem-solving situation.

The small group is a useful tool for studying problem solving and may also prove to be a useful pedagogical format for problem-solving instruction. While the former was supported by this study, the latter was not measured in any manner to determine improvement.

Two directions for future research are suggested by the results of this study. First, studies measuring the small group as a pedagogical format for improving problem-solving ability seems warranted. Second, the relationship between teacher beliefs, student beliefs and student probem-solving ability would be of interest.

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SELF-DIRECTED USE OF DIFFERENT TYPES OF REPRESENTATIONAL MODELS IN MIDDLE SCHOOL PROBLEM SOLVING

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The 'draw a diagram' heuristic is perhaps not enough for middle school problem solvers who are in transition from the concrete operational to the formal operational stage of cognitive development. Since the learning of mathematical ideas is enhanced through interaction with representational models, and problem-solving experiences often include refining, stabilizing, and coordinating mathematical ideas, the self-directed use of a variety of representational models in problem solving was studied. In addition to the familiar concrete, pictoral, and symbolic classification, two others were found to exist. They are direct meaningful, indirect meaningful, and nonmeaningful and active manipulative, passive manipulative, and nonmanipulative classifications of representational models.

This paper was motivated by research on representational models as part of the study Microcomputers and Middle School Problem Solving (MMSPS) supported by the National Science Foundation (Grant No. SED-80-166561, directed by Kantowski and Schultz). Previous reports (Schultz, 1984; Schultz & Hart, 1983) focused on self-directed use of models by below-average, average, and above-average ability middle school students. The purpose of this paper is to present a scheme for classifying representational models.

Models in the MMSPS project were used for nonroutine number theory problem solving where all but one of the 25 problems presented were in the context of some meaningful or hypothetically meaningful situation, as opposed to a mathematically abstract situation. One of the meaningful/hypothetically meaningful problems was as follows: "E.T. was sitting in the closet eating Reeses Pieces. When he started he had 48 Pieces. When he stopped he had 26 Pieces which he put into piles. The first pile contained only orange Reeses Pieces. The second pile contained only brown Reeses Pieces. The third pile was yellow Reeses Pieces and the fourth was green. He noticed he had 3 more orange than brown, 3 more brown than yellow, and 3 more yellow than green.

How many Reeses Pieces were in each pile?" The mathematically abstract problem was: "The perimeter of a right triangle is 24 cm, and the area is 24 sq cm. What are the lengths of the sides of the triangle?"

Representational models provided for use by the subjects were as follows: (a) the real object mentioned in the problem (e.g. Reeses Pieces), (b) pictures of individual objects in the problem that could be manipulated freely (e.g., pictures of bicycles), (c) models of objects mentioned in the problem (e.g., small plastic cars for a problem about cars), (d) models that stand for objects mentioned in the problem (e.g., wooden inch cubes to stand for people), and (e) number tiles (e.g., number cards in a problem about ages).

The organizational scheme of representational models presented in this paper was generated from an analysis and synthesis of student observations, questionnaires, process scores on students' problem solving, student interviews, and self-report data. As was found in a similar interpretation of data in a study by Lesh and Schultz (1983), the refining, stabilizing, and coordinating of mathematical ideas continued during the problem-solving episodes. The organizatinal scheme for representational models to be presented in this paper, then, is likely applicable to instruction in mathematical concepts, facts, principles, procedures, and algorithms, though research should test this out.

THE STUDY

The Questions. First, we wanted to know if there would be any observable pattern in self-directed use of representational models; and, second, if using these models correlated with better problem solving. Finally, we asked what motivates these students to use or not use models.

The Method. The project, which took the entire school year, included all three seventh grade classes of a suburban middle-class Atlanta public school. In all there were 30 subjects out of 78 seventh graders. On the first day of the first phase of the study a General Questionnaire was administered and an

orientation to the project was given. This was followed the next day with a Cold Problem-Solving Test (no prior instruction about problem solving) with representational models available for voluntary use. Then there was a two-week instructional period in which students were given a series of lessons on number theory, heuristics (Polya, 1957; Schoenfeld, 1980), representational models (called "provided materials"), and the microcomputer. The models available were of a variety of types. In addition to those already listed were bottle caps, cups, dice, styrofoam pieces, Cuisenaire Rods, pennies, sticks, and construction paper. The microcomputer was used to present the problem, facilitate hint selection through the use of a menu, and to record student work. A Pre Problem-Solving Test was given immediately after the two-week instructional period, which marked the end of the first phase of the project. (All problems presented were in sets of five to eight problems per set, where each set was for instructional, computer, or testing purposes. Each problem coresponded to another problem in another set according to structure.) The second phase occurred during the next eight months while students individually worked nine problems. Due to the number of students using the microcomputer, they were able to work about one problem a month. A problem followed by a menu of hints was presented one at a time on the computer. Hints varied in picture and symbolic representations, which varied from static to animated formats. Models were available for voluntary use at each of two computer stations in the classroom. The last phase of the study consisted of a day of administering a Post Problem-Solving Test with models available for voluntary use, a day of administering a Post General Questionnaire and an Attitude Questionnaire, and a day of individual interviews with selected students.

Results. The notion that models are for use only by low-ability students was not supported by this study (see also Schultz, 1984; Schultz & Hart, 1983). The average-ability students who were functioning on grade level were more responsive to and more successful with models in their problem-solving sessions than the below- or above-average students. Increased use of models corresponded with increased problem-solving success (determined by a weighted process score for each problem). The critical moment seemed to be after the computer problems were completed and students took the Post Problem-Solving

Test. Interview data revealed that students had become dependent on the static and animated representational models available through the computer hint menu. They eagerly reached for "provided materials" during post testing because they realized that they could do the problems if they had some "help."

TYPES OF REPRESENTATIONAL MODELS

Data showed an uneven attraction toward and usefulness of the "provided materials" which were available throughout the study for voluntary use. This might be explained by Piaget (Flavell, 1963) in his theory of adaptation to a mathematical idea. Adaptation to the mathematical idea through interactions with nonsymbolic representations (e.g., blocks, pictures, dice) occurs simultaneously to adaptation to the mathematical idea through interactions with symbolic representations (e.g., number tiles, equations). The accommodation and assimilation of these two systems to each other is also in progress. Maneuvering an interplay between two representational modes has been shown to be considerably difficult for primary grade children (Behr, 1976) and preservice teachers (Khoury & Behr, 1982). The middle school students in the present study, who were in transition from the concrete operational to the formal operational stages of cognitive development, demonstrated these difficulties. There was a favoring of some types of models over others with a propensity toward direct manipulation, especially after experiencing successful use of static and animated representations on the microcomputer. This seems to validate the assumption (Champagne & Rogalska-Zar, 1984) that manipulating microcomputer representations and manipulating concrete or pictoral representaions by hand are equivalent.

Organization of types of representational models into a Concrete, Pictoral, and Symbolic, classification is an old one. Bruner (1960, 1966) and Dienes (1965, 1967) each spoke to this as it related to the works of Piaget (Flavell, 1963) on cognitive development. A Direct Meaningful, Indirect Meaningful, Nonmeaningful classification is then applied across the Concrete, Pictoral, and Symbolic types. Direct meaningful models were ones such as pennies for problems about pennies, Cuisenaire Rods for problems about

Cuisenaire Rods for problems about Cuisenaire Rods, pictures of running shoes for a problem about runners. Indirect meaningful models were ones such as wooden cubes for a problem about fuel containers and rectangular pieces of construction paper to represent trampolines. Nonmeaningful models might be bottle caps for a problem about odd and even numbers. Though at this time analysis of data across problems is still in progress and conclusions cannot be drawn from that, it has already been determined from preliminary analysis that in general as students increased their use of meaningful materials, they were more successful with problem solving. It should be pointed out that the problems were very difficult for the subjects.

Finally, each intersection of the above classifications can be further classified according to an Active Manipulative, Passive Manipulative, and Nonmanipulative classification. Active manipulation of a concrete model is Bruner's enactive mode of operating. It was found in this study that students' active manipulation of not just concrete models, but of pictures or symbols in the form of number tiles occurred. Manipulation was the attraction more than what they manipulated which seems to contradict the findings of the Threadgill-Sowder and Juilfs (1980) study on manipulative versus symbolic approaches in middle school - a study which also appeared to be testing for differences in manipulative versus nonmanipulative approaches. Barnett and Eastman (1978) found that preservice teachers learned equally well whether they engaged in active manipulation or passive manipulation, where they were shown demonstrations of manipulated materials. The present study suggests that this might also be true for middle school students.

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LINKS AMONG CONCEPTUAL UNDERSTANDING, COMPUTATIONAL SKILL, AND APPLIED PROBLEM SOLVING ABILITY

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Due to the availability of powerful new computer utilities, for the first time, certain realistic applications can be used to introduce a wide variety of mathematical topics. Non-answer-giving phases of problem solving can begin to be addressed even in conceptual areas characterized by complex number-crunching routines or sophisticated underlying conceptualizations. In past research of our Applied Mathematical Problem Solving project, the main way that we were able to focus on non-answer-giving phases of problem solving was to create problems which do not require elaborate answer-giving routines. Now, realistic problems can be addressed in which utilities are used to carry out answer-giving procedures. Even in complex problem solving situations, the student's attention can be focused on conceptualization formulation, trial solution evaluation, and other non-answer-giving phases.

Recently, several research projects (e.g., Lave, 1984; Lesh, 1984; Schoenfeld, 1984), evolving from very different disciplinary perspectives, have begun to converge on a common cluster of observations involving striking mismatches between apparent "in-class" mathematical computational/conceptual abilities and comparable capabilities exhibited in realistic problem solving situations. Furthermore, it is not necessarily the case that performance in realistic situations is worse than in (presumably simpler) in class situations. Some of these results are shedding new light on relationships between "basic skills" and "mathematical literacy in realistic context."

Lave and colleagues, for example, in their investigations of math-related problem solving in everyday problem solving situations, like grocery shopping, have noted that shoppers whose computations were virtually error-free in the market place often scored quite low on comparable items on standardized

tests (e.g., Rogoff & Lave, 1983; Lave, 1984). Similar observations can be found in the work of Schribner and Cole (1981, Carraher and Carraher (1983), and others investigating real world problem solving abilities of average ability individuals.

Our Applied Mathematical Problem Solving project has shown that: (1) The processes needed in the solution of typical "word problems" frequently differ significantly from their concrete/real-world isomorphs (Lesh, Landau, & Hamilton, 1983). (2) Many problem solving heuristics and strategies which have been hypothesized to facilitate in-school problem solving performance often produce counter-productive results in realistic applied problem solving situations (Lesh & Zawojewski, 1983). These results appear to derive, in large part, from the fact that our research focuses on problems in which the question "What should I do when I am stuck?" (i.e., when I am not aware of any relevant ideas to bring to bare on the problem) is less relevant than the question, "What should I do when I have several, perhaps conflicting, ways to think about the problem (or parts of the problem) and all of these conceptions need to be refined, adapted, or integrated to fit the situation at hand?" Solutions to our problems tend to emphasize non-answer-giving phases, and the underlying ideas that our students use nearly always are at intermediate stages of development. Solution processes often are characterized by "local conceptual development."

Recently, our attention has been focusing even more directly on situations in which students are confronted with the need to mold and assess the results of powerful "conceptual amplifiers" (included computer-implemented answer-giving tools). I will say more about this in a moment. For now, I want to mention a result of our past research which makes this tool-using emphasis even more interesting. The result, when reported to me by one of the members of our AMPS staff, came as quite a surprise. Although, it was by no means difficult to explain (or at least rationalize), it certainly was not anticipated. The result concerned "background data" that we had gathered on the 23 seventh graders who had participated in a semester-long problem solving course, which I have described in some detail in numerous other papers. The result was this:

Prior to the "course" (this term, as I will explain later, is somewhat of a mis-nomer), our students ranked slightly below their peers on three sub-

tests of a standardized mathematics achievement test battery. The sub-tests had to do with concepts, computation skill, and problem solving, respectively. After the course, our students regressed even more, compared with their peers, on the sub-tests having to do with concepts and computation. However, they not only increased relative to their peers on the problem solving subtest, they surpassed them in absolute terms.

What explanation can we offer for these non-standard but statistically significant results? What problem solving capabilities had we instilled in our students which did not manifest themselves on "conceptual" or "computational" items? I am not prepared to answer this latter question, but I can address the former.

First, let me point out that our "course" really did not attempt to "teach" anything. Two days per week, three-member teams of students worked period-long problems simulating realistic situations in which mathematics might be used by students or their families. One other day each week was devoted to some type of testing session. During the remaining two days each week, most of the students worked individually on problems of the standard puzzle-games variety. These latter two days functioned primarily as "filler" to occupy most of the students while some individuals were taken out of the class for individual problem solving sessions.

Normally, our students would have been in math class five days per week. Because of their work with us, however, their regular math class was reduced to three days per week. Because it was the intent of the school to devote a substantial portion of the curriculum to "problem solving," our students were thought to be sacrificing only what would otherwise have been "free periods." In spite of this rationale, it came as no real surprise to project staff that our students fell somewhat behind their peers in the areas of concepts and computation. In reality, regular mathematics classes consisted almost entirely of instruction related to computational (and/or what standardized tests call "conceptual") tasks - with the remaining math-relevant time devoted to "word problems" similar to those on the problem solving component of the standardized test. Our students, on the other hand, spent at most three days per week on "concepts" and computation instruction - and they were given virtually no explicit instruction on word problems. Still, the experiences

that our students were given enabled them to outperform their colleagues on problem solving items in spite of relative losses in computational abilities.

We suspect, although our data do nothing to confirm, that the primary things learned by our students from their problem solving experiences were to interpret problems meaningfully and to assess the sensibility of results. The results of others (e.g., Lave, 1984) would suggest that, if computations are interpreted meaningfully, conceptual and computational abilities imbedded in these meaningful situations can be expected to be far superior to that exhibited in other contexts.

Currently, we are doing several things to explore the above kinds of conjectures. At WICAT, Alan Hoffer's algebra project has developed a powerful equation-solving utility which we call a symbol manipulator. Among the things that the symbol manipulator can do is to automatically carry out, in either a single-step "magic" mode, or in a one-step-at-a-time "trace" mode, the answer-giving phases of algebra or calculus problems. This utility can enable us to introduce lesson/unit topics using honest applications that use the to-be-taught skill or concept. In problem solving situations, it also can enable students to escape the computational drudgery of answer-giving phases so that they can investigate non-answer-giving processes and understanding. We believe that these non-answer-giving processes and understandings often make up the "core" of the to-be-learned content in mathematics. Jim Fey's project (Fey, 1984) has described some of the power that this "tool-enhanced" approach can produce at the college level. We are exploring similar instructional implications for middle school and high school.

Currently, Ralph Ottenheimer, one of WICAT's programmers responsible for creating the symbol manipulator, is working on an enhanced symbol manipulator which processes not only numbers and variables, but also unit labels. This powerful utility incorporates features similar to Judah Schwartz's (1981) semantic calculator (SEMCALC). The unit-label processing capabilities provide natural feedback concerning the sensibility of specified procedures. It also provides what John Seely Brown (1983) calls an "audit trail" to enable students to examine the paths they traverse during solution attempts.

Allan Peoples and Mark Bromley, two math/science teachers at WICAT's laboratory school, have created a two-week computer-directed unit of instruction based on Goodstein's Sci Math (1980) materials. Problems for the unit

were taken from SCIMATH and from the NCTM Sourcebook of Applications (1980). Two versions of the unit have been created. One focuses on procedure-oriented instruction translated directly from SCIMATH into a CAI environment. The other consists almost entirely of problems in which the symbol manipulator utility is used to carry out answer-giving solution phases. In this utility-enhanced version of the unit, students only have to "set up" the problems and assess the sensibility of results. They do very little computation. Only one day of instruction is devoted to a "readers digest" treatment of the procedure-oriented instruction that characterizes the SCIMATH unit.

Pilot data suggests that students who have used the utility-enhanced version of the unit perform at least as well on post tests on answer-giving computations, and they do much better on problem solving exercises.

During the 1984-1985 academic year, we will be extending our investigations to involve larger numbers of stuoents, at several test sights throughout the country. We will also have more utility-enhanced instructional units to use as contents for our research.

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BARRIERS TO MATHEMATICAL THINKING AND PROBLEM SOLVING

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ABSTRACT

Emphasis in mathematics education in recent years has shifted from computation to mathematical thinking and problem solving. While researching children's conceptions of common mathematical ideas, the presenter and his colleagues have noted two factors which impede students as they attempt to engage problem solving and mathematical thinking. First, students seem to acquire inadequate perceptions of mathematics and their role in learning it from early experiences in arithmetic. Second, students adopt counterproductive learning strategies which seem to arise as a consequence of the perceptions they hold. These perceptions and strategies are acquired in the elementary school and become fixed and difficult to change. In the end, the student learns to focus attention on matters external to problem situations. These factors need to be rigorously addressed if genuine improvement in mathematics education is to be achieved.

In the early 1970's, the presenter and a colleague (Dr. Stanley M. Jencks) began to study the conceptual constructs children create for common mathematical ideas. This was done via interviews following a period of activity designed to relax and orient the student to his role and the role of the interviewer. It soon became apparent that the majority of children interviewed did not possess adequate referents for the symbols of mathematics.(Peck, 6) (Similar findings have been noted by others. (Erlwanger, 2; Davis, 1)) What was not quite so apparent to the researchers in the beginning was the nature of the perceptions children acquire and the strategies they come to employ as a result. The perceptions are inadequate and the strategies are counterproductive and become fixed and difficult to remediate. It has since become apparent to the researchers that these

perceptions and counterproductive strategies form formidable barriers to mathematics education, particularly to problem solving and mathematical thinking.

To illustrate, some excerpts from an interview are given below. This interview represents those done with students ranked in the upper third of the student population in mathematics skills as measured by standardized tests, past performance and teacher judgement. This student was given the problem 2/3 + 1/4. She performed the operation, rewriting it vertically (which she said was necessary) and proceeding through the usual manipulations associated with the common denominator method of fraction addition. However, these questions remained: (1) Did the student understand what the rule was accomplishing for her, and (2) could the student answer questions about how the symbols relate to reality?

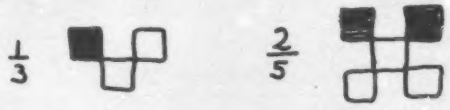
Interviewer: I see what you have done here. Could you tell me which is larger, 1/4 or 3/12?

Student: One-fourth.

Interviewer: Why is that?

Student: Well, here (pointing to 1/4) you have great big pieces, and here (pointing to 3/12) you have twelve little pieces. One of the four big pieces is more than three of these teenie pieces.

When asked to sketch 1/3 and 2/5 and decide which would be more, she did so as shown below:



When faced with the need to explain and being unaware of her misconception, she simply ignored the "reality" of her sketches and returned to her rules (the larger the denominator, the smaller the pieces).

Voice inflections, body language and her words conveyed the impression that this student (lacking proper conceptual referents for the symbols and

operations on fractions) perceives mathematics as a collection of rules whose attachment to reality is vague, at best, and that such an attachment is unimportant. Furthermore, the student does not perceive a necessary underlying logic for the rules—they just are.

This student is not unique but is representative of the "better" students interviewed over the years. Many, like this one, possess manipulative and verbal descriptions of the rules, but lack conceptual referents and cognitive understandings that lend meaning and provide a guide for thinking. The experience base of this large proportion of students (about 90 percent) seems to have been focused upon memorization of facts and rules.(Peck, 6) Their posture towards mathematics reflects this experience. They have come to view mathematics in nonuseful ways by default. The following perceptions seem to be representative of those encountered over the years:

1. Mathematics is a collection of rules which are chained together to provide answers in narrowly specified circumstances.
2. Mathematics is not helpful in solving real problems.
3. Mathematics was invented by geniuses. Most ordinary people cannot be expected to understand it.
4. Right or wrong cannot be decided by the learner, but is the province of the answer key or the teacher.
5. The learner's role is to be told specifically what to do, then follow instructions precisely.

The most damaging aspect of these perceptions lies in the counterproductive strategies which students seem to develop from them. Consider the following exchange with another bright sixth grader:

Interviewer: You need 1/4 yard of material to make a puppet. How many puppets could you make from 2/3 yard of material?

Student: Would you add the fractions?

Interviewer: I don't know. Why do you think that should work?

Student: Oh, I know! You would multiply! Right?

This student, although able to perform the operations flawlessly, could not use them to solve a problem. Her strategy involved dependency and manipulation of others into the problem solving situation. The constant search of the interviewer's face for clues, her attempts to get him to commit to a direction, and subsequent observations in a classroom context revealed a memory based strategy augmented by a manipulation of the social setting.

Most students, like the one described above, adopt strategies that focus on considerations external to the subject matter. Among these are the following:

1. A dependent incorporation of the instructor or trusted peer into the problem situation.
2. Manipulation of the social atmosphere to accomplish success.
3. Avoidance strategies to preclude involvement.
4. Mnemonic schemes, focusing upon form, are used to retain mathematical information.
5. Ritualization of mathematical processes.
6. Attempts to find logic in symbol patterns, not in meanings attached to the symbols.

With conceptual difficulties, inadequate perceptions, and counterproductive strategies a part of most students thinking (less than 10 percent seem to escape), it is not surprising that many do not learn to think mathematically nor to solve problems.(Peck, 6) Most tragically, they learn to distrust mathematics and mathematicians and seek to avoid the subject.

The problems described above do not seem to disappear with maturity but are in evidence in even more strength at the junior and senior high school levels and even at the university. Instead of analyzing a problem, students appear to search the "file cabinets" of their minds for a rule or formula. When memory or learned strategies fail, they are blocked. They are like some of the author's calculus students who could not solve a problem because

they had forgotten the formula for finding the midpoint of a line segment containing (-4,2) and (6,10) as end points. It did not occur to them to simply count. Common sense (in their perception of mathematics) did not have a place in the process of solving problems.

There is much currently being proposed to improve mathematics education. Most suggestions involve more time on task, increasing standards, course work for teachers and other administrative considerations. These will undoubtedly help. However, in order to gain sufficient progress in mathematical thinking and problem solving, the education of prospective teachers (particularly elementary teachers) should proceed with the following in mind:

1. An examination by both professional mathematicians and educators of the conceptualizations, perceptions and strategies children acquire that prevent or hinder mathematical growth and find ways to eliminate them.(Jencks, 4)
2. Alter educational experiences for prospective teachers to help them become problem solvers in their own right and thus prepared to help children learn to solve problems.
3. Alternatives to standardized tests need to be developed which focus on an evaluation of common sense, conceptual understandings. (Jencks, 5) This requires some contact between a child and another knowledgeable person. Analysis, defense of results and conceptual understanding should replace right answers as a measure of progress.
4. Children need to be placed on a footing for the application of their own logic to problem situations. Concrete materials can be used for this purpose.(Jencks, 4) Problems should be developed whose explorations will reveal the central concerns of the mathematics curriculum.(Peck, 7)

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THE EFFECTS OF SPATIAL ABILITY AND PROBLEM PRESENTATION FORMAT ON MATHEMATICAL PROBLEM SOLVING PERFORMANCE OF MIDDLE SCHOOL STUDENTS*

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This study investigates whether the relationship between spatial abilities and performance is affected by the way in which problems are presented, where the problem presentation format systematically varies the strength of the suggestion that the use of a diagram might be helpful.

Subjects were 384 middle school students. The instruments used were the Monash Space Visualization Test (MSVT) and parallel investigator-made 17-item problem solving tests (Pretest and Posttest).

Major findings included: (1) There is a strong correlation between spatial ability and problem solving performance. (2) The lack of a general format effect on Posttest results contradicts the widespread generalization that diagrammatic representations help students understand and solve mathematical word problems. (3) The interaction between spatial ability level and format was highly significant.

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*This paper is based on the author's doctoral dissertation, completed at Northwestern University under the direction of Richard Lesh.

The importance of the relationship between spatial visualization abilities and mathematical problem solving is supported by the pervasive use of graphs, diagrams, tables, and other visual displays in most branches of mathematics. Diagram use is typically presumed to mediate between spatial visualization abilities and problem solving performance. However, the usefulness of a diagram depends on the problem type, the diagram type, and the ability of the problem solver to create, interpret, and modify pictorial representations. While the heuristic, "draw a diagram," is almost universally prescribed as an aid in problem solving (e.g., Charles & Lester, 1982), the conditions under which it is most likely to be helpful have not been examined.

The purpose of this study was to investigate how spatial visualization ability, problem presentation format, and their interaction influence middle school students' performance on mathematical problems which vary in difficulty and in the extent to which a diagrammatic representation is likely to facilitate solution. The term "diagrams" refers to any two-dimensional frameworks for organizing data, as well as pictures, graphs, and schematic figures.

This article deals primarily with results pertaining to the interaction between spatial ability and problem presentation format. Other questions, pertaining to such issues as sex differences and effects of the main variables on performance on specific problem types (more versus less difficult items, items with high versus low inherent visuality) are discussed in Landau (1984).

METHODOLOGY

The first stage of data collection included the administration of a 17-item mathematical problem solving pretest and the Monash Space Visualization Test (MSVT, developed by Clements and Wattanawaha, 1978) on two consecutive days to over 400 students in three middle schools. After reasonable follow-up attempts at make-up tests, scores on both tests were obtained for a total of 384 students, who became the subjects of the study. Scores on the MSVT were used to block the subjects in groups of four, then the four subjects in each block were randomly assigned to the four treatment groups. The second stage of data collection involved a 17-item mathematical problem solving posttest, constructed to be parallel to the pretest based on pilot results.

Three experimental conditions were used to present the posttest set of items. These were intended to suggest, to varying degrees, that diagrams may be useful in solving the problems. (1) ASSESS condition: subjects were instructed to rate all the problems with respect to the usefulness of a picture or diagram before actually attempting to solve the problems. (2) DRAW condition: in addition to rating the problems, subjects were directed to draw and use a diagram in the solution of each problem. (3) SELECT condition: in addition to rating the problems, subjects were provided with two accompanying diagrams for each problem and directed to select one or create their own for use in the solution. In the CONTROL condition, subjects rated the problems with respect to anticipated difficulty before attempting to solve problems.

Students of high versus low spatial ability were expected to respond differently to these conditions, resulting in an interaction. Applying the more general aptitude-treatment interaction compensatory model of Salomon (1971), low spatial ability subjects were expected to benefit most from the SELECT condition, which might interfere

with successful performance on the part of high spatial ability subjects. By providing an accurate representation which a low spatial ability subject may not have been able to produce independently, this format is much more likely to enhance performance.

Means and standard deviations for the three major measures (MSVT, Pretest, Posttest) for all 384 subjects and for the four treatment groups appear in Table 1. The procedure by which subjects were assigned to the four treatment groups guaranteed that they would be equivalent with respect to the MSVT; the Pretest means, while not as much alike as the MSVT means, suggest that the subjects in the four treatment groups had very similar problem solving abilities as well.

Table 1. Means and Standard Deviations on MSVT, Pretest, and Posttest for All Subjects and for each Treatment Group

		MSVT		Pretest		Posttest	
Subjects	N	mean	sd	mean	sd	mean	sd
All	384	15.00	5.35	5.55	3.76	4.98	3.46
Control	96	15.00	5.36	5.54	3.85	4.93	3.50
Assess	96	14.98	5.30	5.54	3.88	5.16	3.76
Draw	96	14.97	5.31	5.67	3.64	4.69	3.16
Select	96	15.04	5.42	5.45	3.65	5.17	3.37

An analysis of pretest data revealed that high versus low spatial ability as measured by the MSVT (HILO) had a statistically significant effect on problem solving performance on the pretest. There were no effects due to treatment GROUP or the GROUP X HILO interaction. Therefore any GROUP main effects or interactions that appear on the Posttest can be attributed to the various treatments.

POSTTEST RESULTS AND DISCUSSION

The strong main effect of spatial ability level on posttest problem solving performance ($p < .001$) was expected since the correlation between spatial ability and problem solving performance has consistently been reported at about 0.6 (e.g., Eastman & Carry, 1975; Guay & McDaniel, 1977). In this study, scores on the MSVT were highly correlated with both problem solving scores (.665 with the Pretest, 0.676 with the Posttest), producing the highly significant spatial level effects. The posttest cell means were 6.98 for high spatial subjects and 2.99 for low spatial subjects.

The format variable GROUP does not, by itself, have a statistically significant impact on problem solving performance. The GROUP cell means show that scores varied in a way which would not support the conventional "problem solving wisdom" present in much of the literature. That is, compared with the CONTROL group (mean = 4.93), the ASSESS group did better (5.16), the SELECT group also did better (5.17), and the DRAW group did worse (4.69). In particular, the finding that instructions to draw a diagram as an aid in problem solving were <u>not</u> generally helpful is noteworthy.

As predicted, there was a significant GROUP X HILO interaction ($p < .004$). The GROUP X HILO cell means, shown in Table 2, reflect the predictions stated above. Compared with CONTROL subjects, HIGHS benefited greatly from the ASSESS condition, while the performance of LOWS was impaired. HIGHS did their worst in the DRAW condition, which was unexpectedly beneficial to LOWS. The best performance among LOWS occurred in the SELECT condition, which was less helpful to HIGHS than the ASSESS condition.

Table 2. Cell means, POST by GROUP, HILO. 48 subjects per cell.

		GROUP			
		Control	Assess	Draw	Select
LEVEL	High	6.77	7.83	6.25	7.06
	Low	3.08	2.48	3.13	3.27

In the ASSESS condition, the potential value of diagram use was made more salient to the subjects. Those with high spatial ability could put this idea to good use, either creating external diagrammatic representations or relying on internal images to help solve problems where spatial ability was relevant. Increasing the salience of diagram use for those with low spatial ability resulted in poorer performance probably because they tried to use diagrams to solve the problems rather than relying on other strategies more appropriately matched to their abilities. It is interesting that such a subtle treatment could apparently have such a marked effect.

For low spatial ability subjects, performance improved substantially in the SELECT condition, as predicted. The DRAW condition proved to be more helpful to low spatial subjects than was expected (though still not as helpful as SELECT). In fact, the rather consistently poor performance of the DRAW group as a whole was usually due to the very negative effect of this condition on high spatial subjects.

A possible explanation for the lower scores of high spatial subjects in the DRAW condition is that instructions to draw pictorial representations were followed without the subjects making judgments about

the appropriateness of the strategy. In the absence of DRAW instructions, many high spatial subjects probably solve problems where visual solution processes are appropriate using mental images alone, quickly recording only the answer. In the DRAW condition, they had to externalize their mental images in a form understandable to others, a time-consuming process which did not improve their performance. This explanation is supported by the data in Table 3. Both the SELECT and DRAW conditions elicited a larger number of omitted items than the CONTROL and ASSESS conditions, presumably due to the greater amount of time needed either to produce a useful diagram or choose between those presented.

Table 3. Mean number of omits per group on Posttest.

Group	CONTROL	ASSESS	DRAW	SELECT
Mean omits*	2.57	2.61	3.79	3.33

*A problem was considered to be an "omit" if there were no pencil or ink markings of any kind in the space provided for work.

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SELECTION OF MATHEMATICAL PROBLEM-SOLVING HEURISTICS: A RESEARCH PROPOSAL

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The appropriate use of any procedure, be it a computational algorithm or a problem-solving heuristic, involves at least 3 distinct steps: (1) recognition that the procedure can be used; (2) selection of the procedure from among available alternatives; and (3) proper execution of the procedure. Thus, an individual may fail to use an algorithm or heuristic even though he or she has acquired the skills needed to use it. With regard to instruction in problem-solving heuristics, what appears to be needed are guidelines for students to enable them to perform the 3 abovementioned steps. This research addresses the following questions: (1) To what problem features do experts attend in determining what heuristics can be used to solve a given problem? (2) After having been trained in the utilization of a number of heuristics, to what problem features do seventh graders attend in deciding which heuristics can be used to solve a given problem? (3) What are the differences and similarities between experts and seventh graders with respect to the problem features they attend to in deciding which heuristic(s) can be used to solve a given problem?

An important ingredient of success in solving mathematics problems is the ability to formulate and carry out a "plan of attack" based on a careful consideration of available information. This involves several cognitive activities, among them comprehension and analysis of information, as well as formation of both mental and overt representations of the problem statement. All subsequent work on a problem is dependent upon the adequacy of the problem solver's comprehension strategies, information analysis, and problem representation. In particular, successful problem solving is a function of the features of the problem statement to which the individual attends.

One of the most perplexing problems of all for mathematics teachers is that it is a relatively easy task to train students in the use of a variety of heuristic

procedures (Charles & Lester, 1984; Lester, 1978; Stengel, LeBlanc, Jacobson & Lester, 1977), but very difficult to teach them to recognize the conditions under which a particular heuristic can be used. Oftentimes, when students are taught how to use a procedure (e.g., a heuristic such as "looking for a pattern") they try to use it when it may be of no use at all (Lester, 1978; Lester and Garofalo, 1982). These students have been taught how to perform the procedure but not when to perform it, how it relates to other ideas, or how to decide what to do when there are alternatives available (Lesh, 1982). This phenomenon suggests that students have not learned to base their choice of actions on relevant structural features of problems. Instead, they rotely attempt to use a procedure that has proven to be successful for them in the past or they attend to irrelevant problem features and choose inappropriate procedures to perform.

RESEARCH QUESTIONS

The appropriate use of any procedure, be it a computational algorithm or a problem-solving heuristic, involves at least three distinct steps: (1) recognition that the procedure can be used; (2) selection of a particular procedure from among available alternatives; and (3) proper execution of the procedure. Thus, an individual may fail to use an algorithm or heuristic even though he or she has acquired the skills needed to use it. With regard to instruction in problem-solving heuristics, what appears to be needed are guidelines for students to enable them: (1) to recognize when a particular heuristic may be useful; (2) to make appropriate choices of heuristics when more than one can be used; and (3) to execute the heuristic correctly.

In order to include use of heuristics as a part of a general theory of problem-solving performance several questions must be answered. This research investigates 3 of these questions, in particular:

1. To what problem features do experts attend in deciding which heuristic(s) can be used to solve a given problem?
2. Both before and after having been trained in the utilization of a number of heuristics, to what problem features do novices (7th graders) attend in deciding which heuristic(s) can be used to solve a given problem?

3. What are the differences and similarities between experts and novices (7th graders) with respect to the problem features they attend to in deciding which heuristic(s) can be used to solve a given problem?

PROCEDURES

Currently, a large collection (at least 100) of process problems which presumably are solvable by one or more of the usual heuristic methods (i.e., guess and check, find and use patterns, working backwards, solve similar problems, solve a simpler problem) is being assembled. The problems will then be given to four problem-solving experts (mathematics education faculty and doctoral students) for solution. Each expert first will be asked to solve each problem in one way, presumably using the method the expert deems most appropriate. After the expert has solved each problem he/she will then be asked to solve each problem in as many different ways as possible. Upon completion, the problems will be categorized according to the heuristic(s) used by the experts to solve them. Note that some problems may be placed in more than one category while some might not be categorized at all. Only those problems categorized with an acceptable level of consistency will be used. Consistency will be determined by the heuristic methods used by experts along with discussions with each expert concerning her/his solutions and the solutions of the other experts. Acceptable consistency will be attained if, after discussion, three out of four experts agree not only that a heuristic can be used to solve a problem but also that it is natural to use it. Only those categories which contain a usable number (at least 12) of problems will be retained. The problems contained in these categories will form the problem set for the rest of the study.

Each expert will be interviewed to determine which problem features (if any) served as cues to: (1) which heuristic(s) would be legitimate and also most natural for solution and (2) which heuristics would be easily ruled out as being useful. Similarities and differences in important problem features for problems within each category and across categories will be sought. Also, similarities and differences in each expert's responses will be noted.

A subset of the problems, consisting of one or two "typical" problems from

each of the problem categories, along with a set of standard textbook story problems and problems with insufficient and irrelevant information will be administered to one class of seventh-graders as a pre-test. A parallel version of this exam will be constructed for administration as a post-test. Solution attempts will be examined and students will be interviewed to determine which problem features they attended to in devising possible plans of attack.

Students will be given instruction in the utilization of heuristic problem-solving methods. Only those heuristics which are represented by categories in our problem-solving set will be included. The instruction will involve presentation, demonstration and explanation of methods and will provide ample opportunity for students to practice applying heuristics with feedback on their attempts. Heuristic methods will be taught one at a time and students will be tested to insure that they can apply each one before going on to the next. Instruction will cease when at least three-fourths of the students demonstrate that they can apply each heuristic to a problem when instructed to do so.

Once it is determined that the students are reasonably competent to use heuristics, an examination of how they decide on when and where each heuristic is applicable can be conducted. A delayed post-test, parallel to the pre-test, will be administered. Students will be instructed to solve each problem any way they can. No mention of any of the heuristic methods will be made at this time. After completion, students will then be given a second post-test consisting of problems along with suggested heuristics for solving each. The purpose of the second post-test will be to determine the extent to which students are able to apply a heuristic when directed to do so. Following both tests, students will be interviewed to determine which features they attended to in attempting to solve each problem.

Several comparisons will be made during this final stage. Students' performance on the pre-test and each of the two post-tests will be analyzed and compared with respect to solution methods, heuristic utilization and cues or features attended to. Comparisons will also be made between students' pre- and post-test performance with that of the group of experts.

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183

THE STRUCTURE OF HEURISTIC PROCESSES

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The heuristic processes "special cases" (SC) and "trial and error" (TE) are examined with respect to four dimensions of analysis which have been used previously to understand the process "think of a problem" (TSP). The network of relationships among these three heuristic processes is explored, illustrating further some of the self-referential capabilities of the "planning and executive control" level in the author's model for mathematical problem solving competency.

It has been suggested (Goldin, 1982; 1983) that at least four higher-level representational systems, or language systems, be incorporated in a realistic model for competency in mathematical problem solving: a verbal/syntactic system for processing "natural" language; a system for imagistic (non-verbal) processing; a formal notational system of representation (with many subsystems); and a system for heuristic planning and executive control, abbreviated "planning language." Output from one system, appropriately transformed, can serve as input for any of the others. In planning language, the unit of analysis is taken to be the heuristic process (Polya, 1962, 1965; McClintock, 1979). Not only can planning language stand in a metalinguistic relationship to the other systems, with the ability to modify some of their rules of procedure, but it also has important self-referential capabilities--heuristic processes can access themselves or other heuristic processes during their execution, and/or act on *domains of heuristic processes* as well as on domains from the other three systems. Had I been asked, when I first began to study problem solving fourteen years ago, to estimate how many different heuristic processes might be needed to understand how people organize their problem solving, I would probably have guessed more than a hundred. I now think there are far fewer--perhaps a dozen

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or two--and that the observed complexity and variety of problem-solving behavior can be understood as arising from the multiplicity of combinations in which these processes can access or act on each other.

To examine the structure of the heuristic process "think of a simpler problem" (TSP), four dimensions of analysis were proposed (Goldin & Germain, 1983): advance planning reasons the solver might have for using the process; specific ways in which the process can be employed in mathematics; the types and levels of problem domains to which the process can be applied; and prescriptive criteria (Schoenfeld, 1979a,b) suggesting to the solver that the process should be used. This paper begins an examination of the heuristic processes "special cases" (SC) and "trial and error" (TE) with respect to these dimensions, with attention to the complicated web of relationships among the three processes.

"SPECIAL CASES" AND "TRIAL AND ERROR"

TE appears to be one of the most fundamental problem solving heuristic processes. Children use TE in many situations before learning algorithmic procedures, and if one thinks of planning language developmentally, TE may well be the first heuristic process of which evidence appears. Likewise, TE may be first from an evolutionary standpoint. While TE and TSP appear very different, SC bears certain resemblances to both: special cases are often simpler than but analogous to the more general situation (thus SC can specialize TSP), while trials can be regarded as conjectured special cases of goal information (thus SC does with given information some of what TE does with goal information). In analyzing the three processes, one finds many of the same specific subprocesses occurring in all. It should thus be stressed that heuristic processes are *organizational units* in planning language--one does not expect to be able to identify the process *governing* a problem solver's behavior from just a brief segment of protocol. Next we turn to our four dimensions of analysis.

Advance Planning Reasons. Some of these are listed in Table 1, with the first column drawn from the earlier analysis of TSP (Goldin & Germain, 1983). While the close parallel between SC and TSP should be noted, the planning reasons in the second (SC) column presuppose that the solver knows special cases are under

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Table 1. Advance Planning Reasons for Employing Heuristic Processes

Think of a Simpler Problem	Special Cases	Trial and Error
(a) solver hopes to *use the answer* to a simpler problem to solve the original or a derived problem by means of another heuristic process or algorithm	(a) solver hopes to *use the answer(s) or value(s)* obtained in special case(s) for another heuristic purpose (e.g. to draw a diagram or graph)	(a_1) solver hopes to *guess the answer* directly and verify it
(b) solver hopes to *imitate the method* that works for the simpler problem to solve the original or a derived problem	(b) solver hopes to *generalize the method* that works for the special case(s) to solve the original or a derived problem	(a_2) solver plans to *exhaust the set of possible answers* until one that works is found
(c) solver plans to *generate a series* of simpler problems, seeking a pattern either in their answers or their solution methods	(c) solver plans to *generate a series* of special cases, seeking a pattern either in the answers, derived values, or solution methods	(a_3) solver plans to *evaluate the outcome* of the trial using one or more problem conditions in order to make a *better* subsequent guess until an answer is reached
(d) solver hopes to *improve understanding, visualization, perspective, "feeling"* for the original problem through construction and solution of a simpler problem, leading possibly to (b)--includes construction of more effective verbal/syntactic, imagistic, or formal representation	(d_1) solver hopes to "see the general in the particular," to *improve understanding, visualization,* or *"feeling"* for the problem through special cases, leading possibly to (b)	(b) solver intends to *test a method* by conjecturing a value for an unknown quantity
	(d_2) solver wishes to *check the reasonableness or consistency* of statements, equations, etc. by verifying them in one or more special situations	(c) solver plans to *generate a series* of trials, seeking a *pattern* in derived quantities or resulting procedures
(e) solver has no particular reason beyond "general principles" [TSP may occur as an unmotivated "trial" when TE is applied to the problem, "find a heuristic process to solve the given problem"]	(e) solver has no particular reason in mind [SC may occur as an unmotivated "trial" during the use of TE on the domain of heuristic processes]	(d_1) solver seeks to *improve understanding, visualization,* or *"feeling"* for the problem by trying possibilities
		(d_2) solver wishes to *check the reasonableness or consistency* of statements, equations, etc.
		(e) solver has no particular reason in mind [TE may itself be a "trial" process]

consideration. The planning reasons for SC should thus be interpreted with greater specificity--e.g. the solver using SC hopes to "generalize" a method, suggesting more about *how* to "imitate" a method than is available to the solver using TSP. The planning reasons for using TE (third column) vary more widely. For all three processes, the reasons listed are classified as follows: (a) reasons oriented toward *answers*; (b) reasons oriented toward *methods*; (c) reasons involving *patterns in a series*; (d) reasons oriented toward *construction of representations*; (e) *default* reasons.

Ways of Use. Partial lists are provided in Table 2, with the first (TSP) column again drawn from the earlier analysis. The third (TE) column is especially incomplete due to space limitations. Note that TSP(a)-(f) can be understood as either accessing SC directly, or as generalizing and then accessing SC. Alternatively, it is possible that a subprocess such as TSP(a) can be carried out through a replacement procedure *without* the solver having constructed a true generalization. Note too that a special case need not always be "simpler." SC(g), for instance, may generate cases with extraneous features that complicate the original problem. Specific suggestions for special cases to try can be carried over to TE--e.g. TE(a_1) or TE(b_1) might access information in SC(a)(i),(ii), or (v), while TE in a geometry context might access information in SC(f). The specific ways of use listed in Table 2 must be augmented with the following. In using TSP, the solver must *verify* that a conjectured simpler problem is in fact analogous to the original, while the use of SC *guarantees* (and specifies) the analogy. In using TE, the solver must label each substitution as a *conjecture* or *hypothesis* requiring testing and ultimate acceptance or rejection; while SC substitutions do not demand this.

Domains of Application. Here the relationships between the heuristic processes become complicated and interesting, highlighting self-reference by planning language. Let H_i denote any of the processes under discussion (i=1,2,3): (a) H_i can be applied to the *original* problem P. (b) H_i can be applied to a problem P' arising when H_j is applied to P [P' = H_j(P)], such as a simpler problem, a special case of P, or a problem obtained when a value is conjectured for an unknown in P. Thus we encounter TE applied to TS(P)--"use trial and error on a problem simpler than P," as well as recursive application of a single process.

Table 2. Ways of Employing Heuristic Processes in Mathematics (Partial Lists)

Think of a Simpler Problem	Special Cases	Trial and Error
(a) substitute a smaller number for a larger one [generalize problem to n, then SC(a)(ii)]	(a) substitute a numerical value for a parameter: (i) 0 or 1, (ii) a small value, (iii) a boundary value, (iv) a value for which the problem is easily solved, (v) a value suggested in some way by other problem information, etc.	(a_1) substitute a numerical value for a single unknown, work through the problem conditions to check the guess
(b) substitute a value for a parameter [SC(a)]	(b) substitute values for a set of parameters: as in (a)	(a_2) after (a_1), use the meaning of the problem (imagistic representation) together with the outcome of earlier trial(s) to decide the direction of the current trial
(c) devise an analogous problem with fewer variables [generalize to n variables, then SC(c)]	(c) when the number of variables or dimensions is available as a parameter, substitute 1 or 2 for it	(a_3) after (a_1) and (a_2), or after (a_1) twice, use the outcome of two trials to decide whether the direction of the second trial is favorable, thus conjecturing a direction for the current trial
(d) devise an analogous problem with fewer dimensions [generalize to n, then SC(c)]	(d) consider an extreme case: (i) values for parameters near 0, $\pm\infty$, boundaries, (ii) physical dimensions very narrow, long, etc. (iii) "trivial" case, (iv) "worst" case	(a_4) detect "overshoot" in (a_2) or (a_3)
(e) take an extreme case [SC(d); or generalize, then SC(d)]	(e) set some or all independent parameters equal	(b) substitute numerical value(s) for one or more unknowns, solve for others, and check problem conditions: (b_1), (b_2), (b_3), (b_4) as above
(f) take a symmetrical case [SC(f); or generalize, then SC(f)]	(f) consider the case of highest symmetry or known special features: (i) triangle equilateral, isosceles, right, (ii) rectangle a square, etc.	(c) characterize "search space" in advance, then use (a) or (b) above, or systematic procedure for exhaustive search
(g) choose a subgoal	(g) use "real-life" cases	
(h) make a change of variable		
(i) make a geometric transformation		
(j) restate the problem to apply to a narrower situation		
(k) drop a given condition		
(l) simplify the notation		

(c) Let $Q(H_j,P)$ be the problem, "Apply heuristic process H_j to P." How H_i can be applied to $Q(H_j,P)$. For instance, TE applied to Q(TSP,P) can be paraphrased as, "use trial and error to apply 'think of a simpler problem' to P." An important way to do this is to *try* the "ways of using TSP" listed in Table 2. (d) Let F(P) be the problem, "Find a heuristic process that solves P." It might seem that F(P) is only a paraphrase of P. But when TE is applied to F(P), the "search space" becomes the set of heuristic processes--a very different outcome from the application of TE to P itself. And so forth.

Prescriptive Criteria. In general, prescriptive criteria can be obtained by making as explicit as possible the conditions under which one or more of the "ways" listed in Table 2 can be readily applied to a problem--but this is often very difficult to do. For instance, SC(a)(iv) suggests, "use 'special cases' when the problem is easily solved for certain values," but recognition of such a property in advance might require a well-established representational system for the particular problem domain.

While space did not permit the inclusion of problem examples or protocols to illustrate the analyses, it is hoped that the reader is persuaded of the utility of this approach for characterizing specific competencies. These can then be formulated as *educational objectives* in teaching problem solving.

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CONCEPT-DRIVEN SOLUTIONS TO ROUTINE STORY PROBLEMS

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In attacking routine story problems, do intermediate grade learners use schemata based on understandings of the meanings of the arithmetic operations to guide their thinking? Performance on a test which attempted to determine sixth graders' understanding of the meanings of the operations suggests that the answer is, "It is doubtful." The conceptual grasp of roughly a quarter of the learners, for either multiplication or division, seemed insufficient to guide thinking when solving story problems using these operations. These data, along with correlational and anecdotal evidence, indicate that a greater curricular effort should be given to the concepts of the operations and to the incorporation of these concepts into schemata for solving story problems.

Commentators on the second National Assessment of Educational Progress (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980) noted students' relatively poor performance on multistep problems and problems involving extraneous data. They suggested as an explanation that too little attention in the curriculum is given to the conceptual underpinnings presumably important for solving such problems.

Anecdotal evidence from interviews we have done (Sowder, Threadgill-Sowder, Moyer, & Moyer, 1983) and work by Noddings (1983) also indicate that at least some children do not rely on a conceptual grasp to guide their choices of operations in solving typical story problems. Rather, some try two (or more) of the basic operations and then choose the answer they deem most reasonable. Admittedly, such an approach can be quite effective for one-step problems involving number sizes for which the student has some number sense. On the other hand, applying this approach to multistep

problems or problems with extraneous data could lead to such a multiplicity of options that a student might easily lose track and fail to consider a correct path to a solution. In any case, such an approach would not be endorsed by anyone who believes that schemata based on conceptual understanding, rather than guessing, should play the major role in solving even simple story problems.

Certainly children cannot use conceptual grasp in solving story problems if they do not possess conceptual grasp. The data here were gathered to study the extent of children's grasp of the concepts of the arithmetic operations, and the relationship of that grasp to performance on routine problems.

PROCEDURES

Subjects. Sixth graders in five schools in two midwestern cities made up the sample, representing a variety of socioeconomic statuses. Sixth grade was chosen since sixth graders have had a long-term exposure to all four basic operations.

The test. A Conceptual Understanding Test (CUT) was designed to yield a measure of the children's conceptual grasp. Twelve randomly mixed items, three for each basic operation, made up the test, with two sample items at the start. Each item consisted of short phrases or a drawing to describe a situation, followed by a question. An example is, "12 rows, 25 in each row. How many?" Students were to choose the correct operation and fill in blanks by the correct operation sign. The children were not asked to compute the answer. Only whole numbers appeared in the items. Not all meanings associated with the operations could be represented in such a short test; in particular, only the common "repeated addition" situations for multiplication and "sharing" situations for division were included. Items were considered incorrect if the incorrect operation was selected, or if an incorrect operand was written. Omitted items were not included in the scoring. The CUT was given as part of the testing in a larger project on problem solving.

RESULTS

The percents of students missing zero, one, two, or all three items for each operation are given in Table 1. Performances on the addition and subtraction items were relatively strong. Note, however, that about one-fourth of the students missed at least two of the items for multiplication, with a similar result for division.

Table 1
Percent in Each Category:
Number Missed by Operation

	Addition	Subtraction	Multiplication	Division
Missed 0	58.5	60.8	43.8	51.2
Missed 1	31.6	31.6	32.2	24.1
Missed 2	7.0	4.7	16.4	12.9
Missed 3	2.9	2.9	7.5	11.8

n = 171 for addition, subtraction, and multiplication, 170 for division.

DISCUSSION

The chances of misparking an item or the unfamiliarity with the format of the test could perhaps excuse a student for missing one of the three items for an operation. Thus, inasmuch as the items are very simple and omissions were not counted incorrect, missing two or three of the three items for a particular operation seems a conservative criterion to use for evidence of lack of conceptual understanding. Under that criterion 24% of these sixth graders lacked a conceptual understanding of multiplication, and 24.7% lacked a conceptual understanding of division. Hence, for story problems involving either multiplication or division, about one-fourth of these students would not have a conceptual grasp adequate to guide their choices of operations. These results are broadly consistent with those observed by Ballew and Cunningham (1982), even though they used a different

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methodology and analysis. Hence, having been observed in different parts of the country, the phenomenon may be widespread and not merely a local or two-city weakness.

Having a conceptual grasp does not assure that a student will use that grasp in solving a story problem. To test the relationship between having and applying concepts, the problem tests of the larger testing were subscaled into scales involving the different operations, one scale for addition problems, one for subtraction, etc. Items on these subscales were scored for correct choices of operation. Correlations between scores on these subscales with the corresponding CUT subscales gave correlations between 0.51 and 0.66, except for a correlation of only 0.28 for a very short addition subscale. (All correlation coefficients were statistically different from zero, at a 0.01 level.) These are respectably-sized coefficients and suggest that having a conceptual grasp bears a positive--but not perfect--relationship to one's applying that grasp.

The data from this study support the assertion in the third National Assessment of Educational Progress report that "...students need a much better understanding of the relationship between problem situations and the operation necessary for finding solutions" (1983, p. 6). An immediate, urgent implication of the data reported here is that teachers and/or textbooks in the intermediate grades should devote more time to the concepts of multiplication and division and their applications. The typical emphasis on fractions, decimals, and percents in grades 7 and 8 makes it unlikely that any lack of mastery of the whole number multiplication and division concepts will be remedied in those grades. Indeed, only a wild extrapolation suggests that students' conceptual grasp of multiplication and division for these other numbers is likely to be even more tenuous, with a consequent further depressed performance on story problems and a further diminution of confidence in dealing with even such routine problems.

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BASIC PROBLEM SOLVING SKILLS AS PREREQUISITES FOR ADVANCED PROBLEM SOLVING SKILLS IN MATHEMATICS AND SCIENCE

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ABSTRACT

A pedagogical framework is proposed for breaking problem solving skills into two groups: basic, and advanced. Some examples of extremely basic problem solving skills are: finding the goal and the givens; maintaining a high level of precision in the use of language and symbols; and forming a chain of inferences in a multi-step problem. Examples of more advanced problem solving skills are: the use of analogies and extreme cases, and conscious strategies of hierarchical planning and working backwards. The distinction is motivated by the following experience with teaching lowest-level mathematics and physics courses to college freshmen: it is difficult for students to use advanced heuristics before they have learned the more basic problem solving attitudes and skills. Some of the skills indentified here have not been included in lists of heuristics in the past, probably because they are so basic.

In this paper a distinction is proposed between basic and advanced problem solving skills. We have found this distinction to be useful in deciding what problem solving skills to emphasize first in a remedial mathematics course at the college level. This perspective should also be applicable to secondary school teaching.

BASIC PROBLEM SOLVING SKILLS

In the remedial mathematics course we now postpone an emphasis on many of the more commonly known heuristics, and instead emphasize some basic skills

BASIC PROBLEM SOLVING SKILLS

I. OVERALL SKILLS AND ATTITUDES

A. Precision in Inference
(Such as avoiding inversions and handling nested relations)

B. Precision in Verbal Expression

C. Precision in Expression Using Written Symbols

D. Engaging in a Cycle of Conjecture, Evaluation, and Correction

E. Use of External Memory to Prevent STM Overload

II. COMPREHENDING AND REPRESENTING

A. Finding the Goal and the Givens (and Keeping Track of Them)

B. Willingness to Draw and Improve on Diagrams

III. PLANNING AND SEARCHING

A. Breaking Off Part of the Problem and Keeping Track of Partial Results

B. Basic Multi-Step Chaining of Operators
(Such as the intuitive use of hill climbing and simple means-ends analysis)

IV. EVALUATING THE SOLUTION

A. Asking Whether the Answer Seems Reasonable

Table 1

skills that we previously took for granted. Table 1 presents a list of these basic problem solving skills. As an organizing framework, we will distinguish here between three major stages in a problem solution: comprehending and representing; planning and searching; and evaluating the solution. The skills in Table 1 are then organized into four groups: Under "overall skills and attitudes" are listed basic skills that are essential in all stages of problem solving. The next three major headings contain skills involved in the three main stages of a problem solution. The basic problem solving skills in Table 1 contrast with more advanced skills often included in lists of heuristics, such as: conscious strategies of heirarchical planning and working backwards, using analogies, extreme cases, special cases, and symmetry arguments. These are important skills, but appear not to be as basic as those in Table 1. The list in Table 1 is speculative and undoubtedly incomplete since it is based largely on observations made during tutoring in our lowest-level college courses in physics and mathematics. However, we can illustrate several of the skills with examples from a problem solving protocol.

EXAMPLES FROM A PROTOCOL

The transcript in Figure 1 contains excerpts taken from a video tape of two students working in our course for college freshmen who have difficult in mathematics. The students are engaged in a pair problem solving activity where the first student's job is to solve the problem aloud. The second student acts as a prompter and critic, encouraging the first student to verbalize her thoughts continuously, and asking for justification of steps in the solutions. In this transcript student S2 solves the following "Days Problem" (due to Arthur Whimbey).

> Days Problem: What day precedes the day after tomorrow if four days ago was two days after Wednesday?

The student answers Thursday rather than the correct answer, Wednesday. In the transcript we see evidence for difficulties with several of the basic

TRANSCRIPT

003 S2: (Reads) "What day precedes the day after tomorrow, if four days ago was two days after Wednesday?" Oh God this is heavy...

006 S2: Four days ago was two days after Wednesday and two days after Wednesday is Friday. So, what day precedes the day after tomorrow? So, we back up four days.

007 S1: Four days from what?

008 S2: Friday. So that's...Monday? --Alright so that'd be Monday. Monday was four days ago before Friday... I'm so confused...

009 S2: Oh God I don't believe this. "What day precedes the day after tomorrow, if four days ago was two days after Wednesday?"...

012 S1: What was two days after Wednesday?

013 S2: Friday. --Okay, I'm going the wrong way. So, what day...I can't do this.

014 S1: Sure you can...

019 S2: What day precedes--it means after- the day after tomorrow. Okay, four days ago was Friday, so Saturday
020 then Sunday, then Monday, Tuesday. Tuesday is four days after Friday, so -- so Wednesday precedes -- Wednesday precedes Tuesday right?...

026 S2: Four days ago was Friday, so that brings you to Tuesday...so the day that precedes the
027 day after tomorrow--so maybe it's Thursday I'm -- I really don't know. ---'Cause --- the day that precedes the day--alright tomorrow--huh...

033 S2: What day precedes the day after tomorrow? Okay, okay, that's right, it's Tuesday. That's four days ago was Friday. -- and what day precedes the day after tomorrow. So, it'd be Thursday...

037 S2: --- Yeah, 'cause you're at Tuesday, and tomorrow is Wednesday, and the day that precedes after Wednesday is Thursday.

043 S2: Ok so, Thursday--is the day that precedes the day after tomorrow. So, Thursday.

Figure 1

problem solving skills shown in Table 1.

Precision in inference. In lines 6 and 8 it appears that S2 is counting days in the wrong direction from Friday. She seems to count backwards in time four days to reach Monday instead of counting forward to Tuesday. Perhaps the word "ago" triggers the idea of counting backwards in time. We call this an inference inversion, and categorize it as one example of a difficulty with precision in inference.

Engaging in a cycle of conjecture, evaluation, and self-correction. This subject shows strengths in some problem solving skills in addition to weaknesses. In line 13, S2 seems to realize that she has counted in the wrong direction and in line 19 she counts up four days in the correct direction. Although it is not clear how she detected her error, she has somehow evaluated her first attempt and corrected it, (perhaps simply by doing the step over again). The process of engaging in a cycle of conjecture, evaluation, and correction requires that the student break away from the common belief that problem solving always consists of recalling a well defined procedure and executing it. S2's willingness to attack the problem even though she has no clear algorithm for solving it is a partial strength. It is partial because she does seem to almost give up on the problem. The encouragement of her partner, however, seems to help prevent her from quitting.

Breaking off "pieces" of the problem, keeping track of partial results, and using external memory. In the protocol, S2 rereads the entire problem and in effect restarts her solution a total of eight times, throughout the protocol. These sections were not included in the transcript excerpts presented here due to space limitations. The original transcript is about three times as long and thus gives the impression of even more confusion and difficulty. The repeated restarting of the solution indicates her difficulty in keeping track of partial results (such as "today is Tuesday"). The simple use of paper and pencil as an external memory for this purpose eludes her during most of the protocol. One source of this difficulty is

probably the fact that students have so little experience with multi-step problems in most of their education.

Precision in inference with nested relationships. In lines 19 and 20 the subject shows a misunderstanding of the meaning of the word 'precedes' which to her means 'comes after.' This is apparently a vocabulary difficulty, not a problem solving or reasoning difficulty. However, Even if we accept her unorthodox use of the word precedes, we find a reasoning error in lines 26 and 27, where she says, "that brings you to Tuesday... so the day that precedes the day after tomorrow- so maybe it's Thursday..." She repeats this argument in lines 33 and 37. Given her assumption about the meaning of 'precedes,' her answer should be Friday, but she answers Thursday. The sub-problem she appears to be working on at this point concerns a set of nested relationships: "What is the day after the day after tomorrow, if today is Tuesday?" She seems to drop one of the "after" relations implied by this question to arrive at Thursday instead of Friday. This is interpreted here as a difficulty in dealing with nested relationships. These relationships are ubiquitous in mathematics. Such relationships are seldom used as often in natural language, and this may be one reason for the difficutly illustrated here.

CONCLUSION

Our experience with problems like the Days Problem is that college remedial students find them challenging and make many errors. I was greatly surprised by this at first. The problem contains no mathematics! All students are intimately familiar with the factual content and subject matter of the problem (the temporal relationships between days of the week) and use them in daily life. The problem is simple enough to make the number of errors surprising, and the students' difficulties on this and other problems indicate that basic problem solving skills like those listed in Table 1 cannot be taken for granted. Due to space limitations only

a few of these have been discussed here. One method for developing these skills using pair problem solving has been implemented by Whimbey and Lochhead (1981, 1982). These skills appear to be more basic than many standard heuristics such as using analogies, extreme cases, simpler cases, symmetry arguments, and conscious strategies of hierarchical planning or working backwards. Teachers should be made more aware of students weaknesses in basic problem solving skills since such an awareness can influence the goals and methods of instruction. Some of the skills in Table 1 have not been included in lists of problem solving heuristics in the past, probably because they are so basic. Most of these skills could appropriately be called basic thinking and reasoning skills. When we attempt to overcome these difficulties, our job as instructors essentially becomes helping students learn to think.

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CLASSROOM ENVIRONMENT

RESEARCH ON THE ROLE OF PLAY AND GAMES IN THE LEARNING OF MATHEMATICS

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This paper presents a critical review of research dealing with the role of play and games in the construction of mathematical concepts. It shows how most of the studies up to now have dealt with a particular aspect of this question, that of determining the benefits of teaching with games. However, these investigations did not separate the effects of play and games per se from that of the teaching involved. This analysis leads us to a discussion of the conditions that might be needed to conduct research on the effect of games by themselves within the context of concept formation.

There is no need to spend a lot of time looking up instructor's manuals and journals in mathematical education, such as "The Arithmetic Teacher", to notice the great number of games which are suggested. Moreover, according to some recent studies (Graeber, Rim & Unks,1977; Baker, Herman & Yeh,1981), a great many elementary school teachers use games in the teaching of mathematics. However, while the use of these seems widespread in mathematics education, the merits of such an approach have not yet been solidly established (Bright, Harvey & Wheeler,1977; Begle,1979) and need to be further demonstrated. It is from the perspective of specific cognitive objectives that we wish to consider here the use of play and games in the learning of mathematics at the elementary level.

Regarding the learning of mathematics at the elementary level, play and games are usually considered to be pedagogical means likely to contribute to the

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development of knowledge and to influence positively the affective components of learning situations (interest, motivation, general attitude towards mathematical tasks). More specifically, a brief survey of the literature on this subject reveals that at the cognitive level, this pedagogical approach has been associated with three main objectives: 1) to maintain and consolidate previously acquired skills by practicing them through games (Peelle,1971; Edwards et al,1972; Bonner,1975; Wausl-Puryear,1975; Hart,1977; Bright et al, 1979,1981); 2) to improve problem solving ability through games requiring the discovery and use of a variety of strategies (Allen et al,1966,1970; Coleman, 1969; Allen & Ross,1974; Gallegher,1980; Goldberg,1980; Fluck,1982); 3) to develop and construct mathematical concepts by means of play and games (Humphrey,1962,1966; Dienes,1963; Ross,1970; Wynroth,1970; Freitag,1974; Bright et al,1980; Derderian,1980; Schoedler,1981; Kamii,1982,1983).

The field being too broad and complex to be dealt with on all three fronts, this paper will focus on the use of play and games in the third of the above mentioned objectives. A critical review of studies dealing with the use of play in the acquisition and construction of concepts will lead to a discussion of the difficulties encountered in this type of research and will enable us to examine other possible avenues of investigation.

PLAY IN THE CONSTRUCTION OF MATHEMATICAL CONCEPTS

In mathematics education at the elementary level, research and studies aimed explicitly at the acquisition and construction of concepts through the use of play and games are still scattered, fragmentary, with little integration between them. In fact, even within the context of their common objective, these studies do not constitute a unified body of research, but rather involve very differing approaches. One of these consists of the research dealing with the use of "active games" in the learning of mathematical concepts.

ACTIVE GAMES are those which require an important part of motor activity (displacement of the child in space, games involving chase and body movements,etc),

in contrast with other games which are considered to be passive. Following the philosophy of educators such as Froebel, one of the principal spokesman for this approach Humphrey(1966,1968,1975), asserts that children will find it easier to assimilate some mathematical concepts and learn them better through motor activity than by more traditional ways. Cratty (1971), another supporter of this approach, goes as far as claiming that any arithmetical operation can be translated into movements and presented in the form of an "active game". The empirical studies undertaken to verify the validity of these claims report positive results (Humphrey,1966,1968; Derderian,1980; Schoedler,1981). While it is likely that "active games" can contribute to the learning of mathematics, the way in which the research has been conducted and reported does not allow any definite conclusion regarding the role of play in the construction of concepts. The first shortcoming is the absence of information regarding the researchers' own interpretation of the concepts involved as well as a lack of explanation as to how these are related to the games used. Another weakness can be found in the fact that the tests selected are better suited to the assessment of skills than for the evaluation of a constructive process (e.g. California Achievement Test). Finally, in "active games", the two variables "game" and "motor activity" are so intertwined that it is impossible to determine their respective effect on learning.

DIENES provides us with a different approach. Even if his experimental work is not very conclusive, it is at the level of conceptualization that his work must be considered a major contribution regarding the use of play and games in the construction of mathematical concepts. For instance, Dienes (1963,1970) distinguishes three forms of play: manipulative play (handling of objects and instructional material), representational play (objects and playmates used in make-believe situations), and rule-bound play. And, according to him, these different forms of play contribute to different stages of the learning process. He suggests that manipulative and representational play are essential in an initial period of exploration ("free play") in which the child can discover various properties and patterns. He views that rule-bound play can then be used as a second stage of the learning process, when games are introduced in order to provide different examples ("multiple embodiments") of an intended mathematical notion.

Dienes has proved to be extremely creative in developing both instructional material and game-like activities. Logic blocks and multibase arithmetical blocks are two well known examples which have been used widely. These illustrate how all the material he has developed has been conceived in order to provide concrete representations of logico-mathematical and mathematical concepts. His game-like activities also were conceived with the same objective. The rules used in these games were intended to be isomorphic to a given mathematical structure, expecting that its various embodiments would lead to its abstraction.

The acquisition of one such structure, the group structure, was one of the objectives of his two experimental studies (Dienes & Jeeves.1965,1970). In these experiments, subjects used variations of a game-like activity whose rules embodied the group structure. Children who could predict the outcome of all the possible binary combinations of a given set of elements were judged to have abstracted the corresponding notion of a group. However, one can discover sufficient patterns in the binary combinations enabling one to predict each outcome without necessarily becoming aware of the group concept. In fact, it is only when most university students are fairly evolved in their studies of algebraic structures that they realize that a table of operations can specify a group, up to isomorphism. The most one can say about these experiments is that 11-year old children can learn to play games whose rules are patterned on the structure of a group.

ROSS (1970) has worked with 20 educable mentally retarded (EMR) children (aged 4½ to 10) over a period of 9 months (100 minutes per week) using games involving number concepts (board games, card games, etc.). By comparing her results with those of a control group which had been exposed to a traditional form of instruction, she found that the experimental group scored significantly higher on her test. Although such research is unquestionably valuable, its experimental design and the lack of details limit its contribution to the question of the role of play and games in the construction of mathematical concepts. Ross does not provide us with sufficient information regarding the games she used, how they were used, the numerical concepts involved in each game, the initial knowledge of her subjects as well as their improvement. But it is her

experimental design which is most constricting, for even if we had details of the experimental group's knowledge, it would provide us with little information about how play and games have contributed to a child's construction of the number concept. No data measuring group performance can provide it. This suggests that a case study approach would be more appropriate for this type of research.

WYNROTH (1970) spent a whole school year teaching mathematics through play and games to kindergarten and first grade children. The mathematical topics involved were the number concept, addition and place value notation. Games such as dominoes, the card game of war, bowling, etc. were ingeniously adapted and organized for the teaching of a specific and precise notion. Wynroth reported that the experimental groups obtained results significantly superior to those of the control groups, but Bright et al (1977) have taken issue with the validity of his statistical analysis. While this research reported clearly on the games, their use, and their mathematical objectives, the other criticisms levelled at Ross' work still apply. Furthermore, the value of this investigation suffers from a common type of weakness: the tests used did not evaluate the child's construction of the intended concepts but merely assessed the subject's acquisition of skills.

FREITAG (1974) investigated the teaching through games at various levels. One of his experiments was at the elementary level and involved six grade-4 students who were taught to play a "function game" for 50 minutes. The investigator reported progress between a pretest and posttest on the notion of function. Such research may indicate that even when games are used for a very short period of time they may contribute to the learning process. However, Freitag does not give us any information about the content of his tests, nor does he report each student's progress. Thus the value of his results cannot be ascertained.

BRIGHT, HARVEY and MONTAGUE-WHEELER (1980) reported on an experiment which must be viewed within the framework they propose for the investigation of cognitive effects of games. This framework contains four dimensions: characteristics of the game, instructional objectives, learner-game interactions (heuristics, strategies, etc.), learner-learner interactions (peer teaching,

competition). In their experiment involving 8 classes of 7th grade students, half of them spent four weeks (2 periods of 20 mins/week) playing games of probability and the other half played fraction games to practice fraction skills previously taught. Since we are concerned with the role of play and games in the construction of mathematical concepts, only the probability groups are of interest to us. On the basis of the pretest and posttest results, the investigators have reported that significant learning of probability concepts occurred. Furthermore, relevant to their interest in learner-learner interactions, they have found that grouping the subjects into homogeneous or heterogeneous play groups of four students (based on their initial knowledge as verified in the pretest) had no effect on learning. However, their experimental design precluded any information regarding the presence or absence of peer teaching and competition.

Lastly, KAMII's work (1982,1983) proves to be most interesting for it suggests the possibility of qualitative differences in the learning of mathematics through games. In two consecutive studies with classes of first graders, she spent the year teaching them games involving additions (such as "double war"). Although in the first study her results were comparable to those of a control group, the results of her second study were somewhat inferior. However, she reports that, when compared to the control groups, the experimental groups demonstrated striking qualitative differences in their spontaneous use of various numerical relationships (e.g. $5 + 6 = 5 + 5 + 1$; $2 + 8 = 8 + 2$, etc). Kamii herself recognizes that the test she used to assess the children's progress was inadequate and did not "capture aspects of knowledge building". Another criticism that can be levelled here, deals with the amount of instruction given in these experiments. For instance, it is not clear how much intended or incidental arithmetical instruction occurred in the initial teaching of the games. Neither can we tell how much explicit or implicit instruction was given by the experimenter and the teacher when they themselves played with the children.

CONCLUSION

On the whole, the research regarding the role of play and games in the construction of mathematical concepts provides us with indications, if not with proof, of a positive contribution. It should be noted that practically all

the studies reported here dealt with a specific aspect of this role, that of using play and games as a way for the teacher or experimenter to introduce mathematical concepts. And this was in line with the underlying philosophy, that such an alternative approach avoided the constrictions, both cognitive and affective, that could be the result of formal instruction. Thus, by the very choice of this global objective, to determine the value of teaching with games, this kind of research did not need to separate the effects of two of the intervening variables, that of play and games per se, and that of the teaching involved. However, this distinction would be essential in any fundamental research trying to establish the extent to which play and games can by themselves contribute to the learning process.

Any such research dealing with the learning of mathematical concepts would have to be envisaged within an epistemological framework. That is, a model providing a detailed description of how a child may construct the notion involved would be needed. While an epistemological approach is not required to analyze a game for its structure and characteristics, it is nonetheless essential if we wish to assess its possible contribution to the construction process. Of course, one would then have to raise the question of how much explicit and implicit instruction is involved in the teaching of the game. Finally, such a context allows for the evaluation of a subject's initial knowledge and continued progress.

If by "continued progress" we mean for example more number facts related to addition, then of course written tests would provide us with sufficient data. However, if we wish to uncover qualitative differences in the child's thinking, in his evolution, written work cannot suffice and one is compelled to resort to a more suitable methodology. To follow the child's thinking will require a case study approach allowing for clinical interviews and continuous observation. To determine the effect of games by themselves, after they have been taught to play the game, we will have to restrict ourselves to observe the children play without adult interference and look for their spontaneous evolution.

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CREATING CLASSROOM ENVIRONMENTS EFFECTIVE FOR LEARNING MATHEMATICS

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Hypotheses that particular mathematics-education research seeks to test, originate in underlying positions either for the improvement of existing school practices and procedures or from positions that hold that school practices and procedures do not achieve that for which they are designed. To that end, research should have relation to schools and feedback from them on actual classroom experience with programs based on it. A description of the positions supporting the research conducted at Rutgers Graduate School of Education on children's learning mathematics as well as a brief account of it will be presented.

Research efforts in the teaching of mathematics are in fact derivative ventures coming as an end product of positions on the nature of mathematics, the nature of learning processes, the role of mathematics in the life of the students and in the mature individual that student will be, the role of the school in creating an appropriate mathematics curriculum and devising the instructional procedures by which it is hoped that curriculum will be effectively learned, and finally the preparation of teachers with the awareness of these relationships to carry them out. Moreover, to have impact and consequence, research must be recognized as the instrument by which any and all of the foregoing positions can be rationalized, improved, and made to contribute to the understanding of them. Thus, what follows is a brief sketch of the positions supporting the research to be described.

Mathematics is considered to be a problem solving activity for problems expressed or expressible in quantitative terms. For these purposes, mathematicians have, over time, invented concepts and procedures to do so. From time to time, they have brought into being entirely new branches of mathematics - Calculus and Probability, for example - for solving particular kinds of problems. Mathematics expressed in symbolic terms is thus a universal language employed by the sciences, physical and social, and

the humanities as well.

Epistemology, the philosophical problem of how it is we know and what it is that is known, is among the oldest of philosophical branches (Glaserfeld, v., 1983). In recent times, philosophical speculation has been supplemented with empirical research from which some verifiable descriptions of the learning scenario have been formulated. In general, these indicate that the learners create their understanding of concepts and process through their activity in constructing the concept or process (Dewey, 1938; Piaget, 1970; Bruner, 1978; Glaserfeld, v., 1982; Cobb, 1983). Such construction appears to be the generalization arising from the recognition of common elements in a series of experiences. The experience of this construction is necessary for the acquisition of understanding. Necessary, also, is the apprehension of the structure of the concept or process, i.e., the relationship among its elements that realizes the whole.

In considering the role of mathematics in the life of individuals, we are faced with a problem. Not all students will become mathematicians or be involved in activities that have a vocational need for mathematics. Yet there is certainly a need for expertise in arithmetic by all individuals. Nevertheless, the physical and cultural environments and current reports on them are often expressed in mathematical terms and this creates the need for mathematics if these environments and the reports on them are to be intelligible to the citizen. This need is not necessarily satisfied by high school or college courses in Algebra, Geometry, Trigonometry, Calculus, Statistics, Probability and the like. Indeed, adults other than mathematicians and vocational users of mathematics who have had these courses rarely have occasion to use their knowledge after graduation. Yet they most emphatically need an understanding of mathematics. This presents mathematics educators with the problem of identifying which concepts and what processes in mathematics will serve these needs.

This problem is, of course, directly related to school curricula in mathematics. If the present curriculum is too dificult for some students so that they do not actually have the courses and those that have them do not use what they have ha

the curricular problem for schools becomes the organization of a curriculum that will serve the need. One possibility is a curriculum organized on the basis of how does mathematics get to know what it knows, i.e., mathematics as a way of organizing the world. In any event, the development of a mathematics curriculum appropriate for Americans is a problem facing mathematics educators.

Instructional practices are the mechanism for achieving the understanding that is the goal of the curriculum by methods that are consistent with what is known about learning processes. Understanding is here taken to be such mastery of concepts and processes as readily recognizes the need for both in the context in which they are encountered, the ability to employ them appropriately and skillfully, and finally, to build on them and associate them with other concepts and processes.

Central to the achievement of the goals of mathematics education is the preparation of teachers capable of understanding and addressing the problems of mathematics education. As a minimum this involves a base of knowledge of mathematics on which there can be built the recognition of the role of mathematics in the lives of Americans in consequence of which a curriculum appropriate fo fulfilling this role is organized. Such a program should be supported by intensive study of learning processes and their different manifestations in children and adults and complicated by the large range of individual differences among students.

Mathematics is a system of ideas and procedures that really have little or no relationship to learning processes, the differences between children and adults, the socio-economic circumstances of the students or any of the host of factors that might and do affect the learning of mathematics. The historical order in the development of the several branches of mathematics does not necessarily coincide with an ascending order of difficulty or complexity although traditional school curricula usually follow the historical order. Either in historical order or within areas, the logical order in both instances does not coincide with the cognitive development of children.

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Moreover, the integrity of the several branches of mathematics is often rather scrupulously respected in school curricula although there appears no inherent reason to do so. Early attempts in the seventies (Fehr, 1974) to organize it in terms of a "unified mathematics" were directed for only the brightest of students. This led to an increasingly abstract presentation of the subject making it difficult and in some instances impossible even for the brightest of students to master it. More recent efforts to prepare a "unified mathematics" program for "average" students still seem to ignore the need to make it less abstract.

The general objective of research in mathematics education at Rutgers Graduate School of Education has been to move toward the creation of the classroom environment that will promote the learning of mathematics as it has here been described. It is based on the realization that by and large, but certainly with distinctive exceptions, mathematics is taught without the integration of the nature of mathematics and the nature of learning. Definitions, rules and procedures are illustrated by teachers and practice exercises, precisely the same in form and different only in the numbers used, are provided. Even when, from time to time, some concrete models are provided, the failure to create an environment that permits children to construct the concept involved results only in a kind of mimicry of the definition, process, or idea that does not lead to understanding. Moreover, existing textbooks and mathematical materials generally, not always, are in a format that gives teachers no option but to employ these methods.

The research, therefore, involves the teaching of concepts and procedures in a problem-solving context with problems representing several different embodiments of the concept or process to be learned in a sufficient variety of ways to accommodate different modes of learning in children. In effect, the effort is to create an environment so that the student "does" mathematics much like the way mathematicians do.

During the last several years, efforts to work directly with schools resulted in associations with four schools that led to cooperative projects. The research associated with these projects will be briefly described.

The First major project, The Rutgers University-Middlesex Public Schools Math Project, included several studies in an extended program of coordinated research designed to include many measurements of the performance of the same population from various points of view and at various points in time over a two-year period with 12 to 14 year old children (Maher, O'Brien, Heller, Smith, Normandia, & DeStefano, 1982). For them, tasks were developed that embodied the structures of particular mathematical concepts. A primary objective was to determine whether a correlation existed between success on these tasks and achievement in problem solving. One interesting result of these initial studies indicated quite strong associations between success with group properties and proportional logic tasks and success in solving percent and ratio and proportion problems commonly found in mathematics curricula for students of this age (Maher & DeStefano, 1983).

This led to a second set of studies in a suburban school of Trenton employing similar materials and procedures but with some modification introduced in the light of experience addressed to children of ages 11 to 13. One study included a teaching experiment designed to investigate the ability of students to generalize the solutions to problems by expressing them in terms of variables (Maher, Gnepp, Alston & Ginosar, 1982). This study differed from earlier ones in its attempt to investigate the process by which students arrive at their solutions. Analysis of data is still in progress and involves rather elaborate coding procedures for recording the transcriptions from audio and video tapes of childrens' conversation and behavior as they responded to clinician's questions. Some of the analyses of these data support the findings of the earlier study. In particular, the analysis disclosed that performance on the group properties task was again a reliable predictor of problem solving success.

This relationship raised its own doubts. Was the group properties task in fact a problem solving activity so that the results merely established that success on one problem solving activity was related to success on another problem solving activity? Or, alternatively, were students successful because they already had that understanding of the concept that permitted them to be success-

ful on the task and therefore also had the ability to extend it to the requirements of the problem? Once again, this led to further study with a more complex set of goals with children from a private school in Princeton. The modifications now included a new set of embodiments of the structure of the group properties in concrete, symbolic and abstract form for children a year younger, on average, who were given the tasks as problem-solving activities with two, three, and four children working together (Maher, Alston, & Bakum, 1983). The objectives here were to investigate whether these concepts could be made accessible to even younger children; whether the children could detect the isomorphism in the two different embodiments of the concept; and whether they could identify the concept in the problems they encountered in their classrooms. The results provided support for the view that the children were in fact identifying the structure of the concept and recognizing it when it arose in their regular classroom work in mathematics.

A subsequent study investigated the question of how different embodiments affect childrens' learning processes and for this purpose a new task was developed (Maher, Goldin, & Alston, 1984). Analyses of the childrens' responses disclosed that different embodiments did in fact seem to influence understanding of the underlying concept.

One finding common to all the studies was that all children routinely employed all forms of the embodiments available and moved freely back and forth among them. The implication of this latter finding is that instructional practices should, invariably, include a variety of forms of embodiments to provide for what may be a universal behavior in children of a range of ages and ability. In turn, this requires that the training of teachers include this feature.

The research continues with further hypotheses generated by these studies. A recent association with a local K-8 school in New Brunswick designated as a theme school for the learning of mathematics, science and computing provides the potential for continued research with children and teachers in a regular classroom environment.

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CONCEPTIONS OF MATHEMATICS

THE PERCEPTION OF MATHEMATICS AMONG ELEMENTARY SCHOOL TEACHERS

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In order to evaluate the perception of mathematics the teachers have, we developed a new tool called "Questionnaire sur la perception des mathématiques". The teachers are given three different sets of statements presenting different possible characterization of mathematics: A- mathematics seen as a set of skills; B- mathematics seen as logic and rigour; C- mathematics seen as a constructive process. Teachers are first asked to order these three sets according to the importance they assign to each one and then to quantify their classification, giving marks to the three sets. We tested this tool with a group of teachers enrolled in an in-service course in mathematics education. The results obtained showed that teachers in this experimental group had a more constructivist perception of mathematics than teachers in a control group we have tested altogether.

If we consider the "world of mathematics", we might perceive it in different ways: three of these perceptions seem particularly important if one thinks about school mathematics:

- the traditional perception, which is found among a majority of people, usually those who are not in the field of mathematics. Mathematics is seen from that perspective as a collection of rules, these rules being considered as tools which give the user some power over reality. But this power is somewhat mysterious and magic, and thus a bit frightening. People feel that they cannot control it because, while they see the results it brings, they don't understand how and why it works. In the schools, this traditional perception appears when mathematics is limited to rules and calculations i.e. when the "skills" component dominates if not eradicates the other components of real understanding;

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- the formalist perception which comes out of what has been, for a long time, the dominant philosophy in mathematics; ultimately, mathematics is seen from this perspective as the science of formal structures and of rigourous logic. The mathematical content, the meaning are relegated to what is called meta-mathematics. In schools, this perception appeared, mainly during the "modern math" era, in the emphasis on unifying mathematics around a few basic concepts (sets, function...) and on the research for rigour in langage and symbolisation. The early insistence on these formal aspects has often contributed to their lack of meaning for many people;
- the constructivist perception, not in the mathematical sense but in the epistemological one. According to this perspective, one gets a piece of knowledge if and only if one has reconstructed it for oneself: thus, knowledge is no longer an object to be transmitted to but an object to be built by the learner. If one wants an efficient teaching of mathematics, one should see mathematics as a field of knowledge similar to other fields of knowledge. And one must agree, as Revuz (1980) says, that for the child who is learning as well as for the mathematician who is creating it, mathematics is something to do and not only to look at. Thus, we must give importance to intuition and to all elements which play a part in the building of real understanding.

A TOOL TO EVALUATE PERCEPTION.

We developped a tool to evaluate the presence and relative importance of these three perceptions: this tool, called "Questionnaire sur la Perception des Mathématiques" can also be used to evaluate changes in someone's perception of mathematics. The "questionnaire" proposes three sets of statements, each set characterizing one of the perception already described. These sets are:

A- mathematics seen as a set of skills (traditional perception):
Doing mathematics is - doing calculations;
- using rules;
- using procedures;

- using formulas;

B- mathematics seen as logic and rigour (formalist perception):
Doing mathematics is - writing rigourous proofs;
- using a precise and rigourous langage;
- using unifying concepts;

C- mathematics seen as a constructive process (constructivist perc.):
Doing mathematics is - developping thinking processes;
- building rules and formulas from experiences on reality;
- finding relations between different notions.

People are first asked to order these sets according to the importance they assign to each one, using the symbols = and/or > : for instance, one can answer C = A > B. Then, they have to quantify that classification, by giving marks (for a total of 30) to the three sets of statements: the highest score is to be given to the set that is classified as the most important and the lowest to the least important one. For example, one can give 12 each to A and C and 6 to B.

EXPERIMENTATION AND RESULTS.

This tool was used with an experimental group (n = 18) of teachers enrolled in an in-service program at the "Faculté d'Education Permanente" at the University of Montreal. These teachers were taking a 45-hour mathematics education course in which a special approach was used. This approach, described by Bergeron et al. (1981) and by Herscovics et al. (1981), was based on initiating the teachers to concept analysis according to a model of understanding. This approach was expected to change the teachers' initial perceptions of mathematics. We used the "questionnaire" immediately before and after the course to see if there were any such changes. The same tool was used in similar conditions with a control group (n = 16) consisting of teachers enrolled in the same program but taking a course in a field other than mathematics or mathematics education.

The following table shows the results obtained.

MEAN OF THE MARKS GIVEN TO EACH SET OF STATEMENTS.

group \ perception	A (traditional) pre	post	B (formalist) pre	post	C (constructivist) pre	post
control (n = 15)	7.5	10.4 (+2.9)	9.5	9.8 (+0.3)	13.0	9.8 (-3.2)
experimental (n = 18)	10.8	9.5 (-1.3)	6.3	7.5 (+1.2)	13.7	13.3 (-0.4)

This table is based on the marks given and neglects the classification: the reason for this is that for nearly half the teachers, there was a contradiction between their first classification and the marks given afterwards. For example, on one answer sheet we had C > B > A and C = 10, B = 15 and A = 5. Fortunately, a large number of teachers had made remarks showing that the marks given were the thing to consider as they were not very familiar with the use of the symbols > and = when more than two variables were to be classified. Only one person – in the control group – was rejected because her answer in the postest was impossible to interpret.

Both groups have changed between the pre and the postest. This is not very surprising since most teachers were confronted to the question of the nature of mathematics for the first time in their life, and the pretest had the effect of starting them thinking on the topic and eventually changing their view. However the change was not at all the same for the two groups.

In the pretest, both groups have quite similar means for set C to which both give the most marks but they differ for sets A and B, the control group giving more importance to rigour (B), and the experimental group to skills (A).

In the postest, the control group has clearly evolved towards a balance among the three perceptions, all of which were given almost the same importance (~10): this is not surprising since in the sets characterizing each perception, there are statements whose importance cannot be denied.

On the other hand, the experimental group remains clearly convinced of the

dominant importance of set C while tending to equilibrate the marks given to sets A and B.

This may be a direct consequence of the course followed by the teachers in this experimental group. The focus in that course was on the way children build their understanding of mathematics, thus reinforcing indirectly a constructivist conviction which was less firmly held by the teachers in the control group. The latter, after a period of thinking between the pre and postest were not able to give a hierarchy.

CONCLUSION

To sum up, two elements seem important:

- the tool described here allows an evaluation of the different perceptions of mathematics held by teachers;
- however, these perceptions are not as clearcut as one may think: a teacher who prefers one perception will still consider important some elements of the other ones. This explains why evolution in this area is slow and remains hard to mesure.

Let us add that this tool was used together with a few others: we were looking for convergence of indications obtained from each of these instruments. The reality we are studying is difficult to catch with only one tool but the simultaniuus use of three different tools validated the observations described here, thus making our conclusions more convincing.(1)

(1) The complete results obtained from these instruments should be published soon.

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THE SKILLS-UNDERSTANDING DILEMMA IN MATHEMATICS EDUCATION

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There exists in psychology a reductionist approach to cognition according to which understanding is limited to the mastery of skills. Another point of view is that some psychologists and educators refuse to accept the mastery of skills as an indication of understanding since these skills can be learned by rote (without any understanding). These two extreme positions provide us with a false dilemma, for indeed they are not the only alternatives since a constructivist approach offers another perspective in which the acquisition of skills becomes part of the construction of understanding. In this communication we wish to show how such a constructivist approach can be applied to the two main mathematical activities encountered at the primary level, that is the construction of conceptual schemas and problem solving.

e ability to distinguish between "skills" and "understanding" proves to be
major importance in mathematics education. The teacher who perceives un-
rstanding solely as the mastery of skills is likely to direct his instruc-
on towards drill and practice. On the other hand, the teacher who considers
at understanding is completely divorced from skills cannot but evaluate it
terms of facts and definitions which could very well have been learned by
te. For instance, in evaluating the learning of addition, the first one
ll be interested in knowing if the child has mastered the procedure he
ught, while the second one will concentrate on the memorization of addition
cts.

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If the first teacher's perception of mathematics is essentially algorithmic, the second one's is mostly formalistic. Both approaches could be quite sterile since rules, definitions and procedures are not applied in a vacuum, but in the construction of conceptual schemas and in problem solving which constitute the two principal mathematical activities at the elementary level. However, the question remains, can we and should we always distinguish between skill and understanding for each of these activities.

SKILLS AND UNDERSTANDING IN THE CONTEXT OF CONCEPT FORMATION

Within the context of concept formation, some of the questions implied in the introduction acquire a new dimension. For instance, we can ask ourselves what is the meaning of the word "skill", can we actually construct a conceptual schema without using mathematical procedures, or whether understanding a mathematical concept can be reduced to the mastery of certain procedures? In order to answer these questions, let us take as examples the child's construction of number and of early addition.

The child may very well master the notion of one-to-one correspondence between sets of concrete objects without necessarily involving the concept of number. Most researchers agree that it is only through a counting procedure, when the child applies a one-to-one correspondence between a set of concrete objects and the number word sequence, that the notion of number emerges. Similarly, the addition schema can only arise through an addition procedure, the most elementary one being that of joining two sets of objects and counting them all. These two examples seem to indicate that it is difficult to conceive how the construction of mathematical concepts could be achieved without appropriate mathematical procedures. One might infer that this is only true for arithmetical notions and not for geometric ones. After all, even a young child can recognize a square. But this does not mean he can generate one by himself. To do so he will need some procedures enabling him to compare the four sides and the four angles.

Even if one agrees that a mathematical concept cannot be constructed without procedures, the question remains whether or not understanding can be limited

the acquisition of such procedures. As mentioned earlier, mathematical cedures should not be introduced without a context both meaningful and rel- nt to the child, for otherwise these end up as "rules without reason" mp,1976). "Meaningful" is used here in the sense of related to the ld's intuition which, in the case of number refers to the notion of quan- y (few, many, the same as), for addition it refers to the two action- nas of adding to and joining, while for the square it means visual recog- ion. While being related to intuition, these procedures must also be pre- ted in the context of situations in which the child can perceive a problem, order to become relevant. For number and addition, such problems must e the question of "How many?". For the square, well before the child will it to find areas, its relevance can be established very early through blems of symmetry and tesselation.

when procedures are introduced in a meaningful and relevant way, their uisition does not constitute the end product. For, within the context of cept formation, the process of understanding does not end there. Indeed, mastery of the procedures leads the learner to interiorize them, and re- cting upon these procedures may enable him to anticipate his results and cover mathematical invariants. For instance, we are all familiar with the ld who can count up a set of objects but who does not conserve number. Or, the case of addition, the child who has mastered the procedure for addition who still believes that "seven and one" will give him more than "four and " (Piaget & Szeminska,1947/67). Finally, for the square, we can find many ng children who can draw them but who will not accept one standing on a tex as being a square. All these children have mastered their procedures have not as yet achieved abstraction of these concepts.

achievement of abstraction is not the end point in the construction of erstanding. The need to communicate our mathematical ideas imposes the rning of a highly condensed and efficient notational system and requires t mathematical notions be enclosed into precise, formal definitions. But s process of formalization introduces additional cognitive obstacles. For mple, the child may perceive the equivalence of sums using concrete objects still fail to accept its symbolization $7 + 1 = 4 + 4$. This may be due

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to the equal symbol being interpreted as a signal calling for "an answer" (Erlwanger & Bélanger,1983). In the case of the square, one might be tempted to believe that the child who can produce a correct definition has understood the concept. However, the same pupil might fail the test of abstraction mentioned above. Thus, the definition of a concept can only be considered as additional understanding if prior abstraction has occurred.

The examples we have used enable us now to clarify what we mean by "skills" in the context of concept formation. Very generally, the acquisition of skills results in "knowing how to do", how to accomplish certain tasks. But in the construction of concepts, where can this apply? Referring to our examples, the child "knows how to" count when he has mastered the counting procedure and "knows how to" add when he can join two sets and count all. Thus the acquisition of skills can be viewed as the mastery of appropriate procedures. The role of procedures in the process of understanding has been taken into account in a constructivist model of understanding suggested by Herscovics & Bergeron (1983,1984). In this model, procedural understanding deals with mathematical skills as "know hows" and constitutes a second level of understanding which follows a first level, that of intuitive understanding, and which precedes two other levels, that of abstraction and formalization. In this perspective, skills being interpreted as procedures prove to be essential in the construction of conceptual schemas, but make up only one of the four stages in the construction of understanding. It thus becomes evident that "skills" and "understanding" are not in any conflict and that this model brings out their complementary roles by distinguishing between procedural and non-procedural understanding. We will now examine if this complementarity is also present in the problem solving process.

COMPLEMENTARITY IN THE CONTEXT OF PROBLEM SOLVING

Without dwelling on a definition of what constitutes a problem, we can nevertheless distinguish between a "problem" and an "exercise". In mathematics, an exercise involves a task for which a procedure is known by the student and used by him for practice. This is not the case for a real problem. Indeed, although we assume that the solver has in hand the knowledge needed to solve

, the solution cannot be apparent right from the start or else there simply no problem. Thus, problem solving calls for more than merely applying some ocedures. According to Polya's model (1962), the problem solving process nsists of the following four stages: understanding the statement of the oblem, conceiving a plan for its solution, executing the plan, verifying and sessing the answer. This fourth stage deals with questions such as:
an you verify the result? Can you verify the reasoning?
ould you get the same result in a different way? Can you see it at a glance?
ould you use your solution or method to solve another problem?

e last two questions touch upon what we consider to be the essence of problem lving: that of restructuring the data in relation to the problem, and that of neralization to a class of problems. For instance, everyone is familiar with e problem of finding in two weighings which one among 9 chips is the lighter s. Does the mere fact of finding a solution to this problem imply that one s grasped the rational basis of the solution? Of course not, since a solu-on might have been found by trial and error without necessarily perceiving s relation between the given data and the problem. Most of us are naturally clined to start off with four chips on each side of the scale which will lead an impasse. To get out of it, one has to break the mind set induced by the ile which focuses attention on the two sets weighed, and to truly restructure e data in relation to the problem. The restructuration needed here is to div-e the chips into three equal subsets and to become aware that each weighing cates the lighter chip in one of these subsets, thus eliminating the other two.

e last of Polya's questions, as to whether or not a solution or method can be d to solve another problem, aims at encouraging a process of generalization ch is most important since it leads to solving classes of problems rather n individual ones, and since, furthermore, such a generalization cannot be ieved without prior restructuration. For example, to someone who may have mbled upon a solution of the 9-chip problem without perceiving its rational is, the mere setting of the analogous problem involving 27 chips and 3 weigh-s may trigger off the needed restructuration. This further illustrates how process of generalization can change one's perception of a problem from ing it as an isolated case to that of representing a whole class of problems.

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Indeed, we believe that this process of generalization can prove to be so enriching that we consider it as a fifth stage in the problem solving activity.

Within this context of problem solving, can one distinguish between skills and understanding? Just as we often use the expression "Mark has a good understanding of addition" to refer to his global grasp of this notion, we also say "Marie is a good problem solver" referring to her general attitude and success at solving problems. However, such statements do not enlighten us about the role of skills and understanding in each of these activities. This role might become more tangible if we contrast with our interpretation of "skills" as the "mastery of appropriate procedures". We can then ask ourselves what kind of understanding, procedural or non-procedural, is involved in each of the five stages of problem solving.

To understand the statement of a problem implies understanding the data and their interrelationships, as well as the understanding of the question asked. Understanding is also involved at the second stage, the conception of a plan. Of course, we are staying in the context of a problem, not an exercise, and this means that the solution is not obvious to the solver. Thus, understanding here implies the organization of the given data in relation to the problem into a plan of action aimed at solving the problem. It seems that the understanding involved in these first two stages is non-procedural. In fact, no procedure leading to the understanding of the statement of a problem or the elaboration of a solution plan exists. At most can we think of the use of heuristics, that is, applying some helpful forms of reasoning (e.g. by analogy), using better forms of representation (e.g. make a drawing), and more generally, raising relevant questions (e.g. What am I looking for?). But in no case can we speak here of procedures.

As far as verifying one's reasoning, obtaining a result which is not a solution to the problem signals a deficient plan. On the other hand, a correct solution is no guarantee since it could accidentally be produced by false reasoning (e.g. finding the area of a 6 x 3 rectangle by adding the four sides). To verify the validity of one's reasoning one has to apply it to a related or more general problem and no known procedure exists for doing so. Similarly, the

generalization process involved in problem solving being specific to each class of problems, this negates the existence of any procedure leading to it.

It is in the third and fourth stages, the execution of the plan and the verification of the result, that skills, in the sense of acquired procedures, are involved. The execution of a plan conceived for the solution of a problem requires its eventual operationalization, that is the translation of each step of the plan into an appropriate procedure. For how else can one follow through with the objective of the plan, that of solving the problem, without acting on it, without carrying out these procedures. Of course, to execute a plan implies that it has already been brought down to steps which can be'executed'. Whether this operationalization process belongs to the elaboration or to the execution stage of the plan is not always clear-cut. Procedures are also involved in the verification process. Quite often, the procedure consists in verifying if a result satisfies the conditions given in the statement of the problem . But this does not always apply, as in the case of problems involving the search for a strategy (e.g. the weighing problem) where the verification phase coincides with the execution of the plan.

By way of conclusion,we think we have shown that in a constructivist perspective of concept formation and problem solving, there is no dilemma between skills and understanding. In fact, when the mastery of skills is viewed as procedural understanding it proves to be complementary to non-procedural understanding in both mathematical activities.

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COMPUTER-BASED INSTRUCTION

HOW TO USE A COMPUTER TO TEACH MATHEMATICAL CONCEPTS

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The formation of mathematical concepts is conditional on the learner's active involvement with the subject matter during the process of learning. Theories of learning indicate that the required mental activity on the learner's part can be stimulated by suitably structured (re)presentations of the concepts to be learned. These structures are most easily implemented on a computer using the idea of a microworld: an environment whose objects the learner transforms by his actions.

BACKGROUND

e of the major concerns of mathematics education in recent years has been e formation of concepts by students of mathematics, be it the concept of the mber 5 as generated by a young child or the concept of a Galois group as eated by a student of advanced algebra. There appears to be a consensus ong the research community that the active involvement of the learner with me representation or model of the concept to be acquired is a necessary ndition for his ability to build the mental structures which imply an derstanding of the concept.

veral theories have been proposed for describing and analyzing the process understanding and thus concept formation. E.g., Dienes' learning theory 963) is based on his Constructivity Principle which states that

> " ... construction always needs to precede analysis; with children there is usually a preponderance of constructive over analytical thinking ... " (p. 156).

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The structure of the learning process is summarized by Dienes in the Dynamic Principle, which states that

> " ... learning proceeds in cycles which follow one another in regular succession. Each cycle consists of approximately three stages: a preliminary and rather unstructured 'play' stage, a more structured intermediate constructive stage, followed by the insight, then an anchoring stage in which the new insight is more firmly fixed in place." (p. 156).

More recently, Skemp (1979) has analyzed understanding on the basis of his theory of intelligence. He distinguishes two modes of thinking (intuitive and reflective) at three different levels of understanding: instrumental, relational and logical. Skemp shows that there is no point in presenting a student with an argument requiring a logical level of understanding with respect to a particular concept, if the student has not previously reached at least the level of relational understanding with respect to the same concept.

Using a more concrete approach, Biggs and Collis (1982) perceive structural complexity as the key to understanding. Their SOLO Taxonomy

> " provides a measure of the quality of assimilation in terms of progressive structural complexity." (p. 164).

The levels they use to characterize student responses carry some resemblance to Skemp's levels of understanding. They are labeled prestructural, unistructural, multistructural, relational and extended abstract. In agreement with Dienes' Dynamic Principle, Biggs and Collis state that

> " ... the concepts of SOLO apply again and again to each new learning episode." (p. 214).

The similarity between the three theories is striking. All three stress that learning proceeds, for each concept and concept, in stages, the first stage being an unstructured groping, a trial to familiarize oneself with the objects under consideration and the actions which may be applied to them. In the following stages greater familiarity with the objects and actions increasingly allows the learner to observe regularities and thus structure the domain. It

s implicit, if not explicit in the three theories that it is the learner's
n actions which allow him to construct his mental representation of the
oncept, eventually reaching the highest level according to the respective
heories.

t is instrumental for this process of familiarization and structurization
hat appropriate means are placed at the disposal of the learner. These means
ust represent the concepts to be learned and they must be appropriate in the
ense that

> " ... models are useful as teaching aids if they have some heuristic efficiency, i.e., the capacity to stimulate the process of reasoning and permit its progress by their own means." (Fischbein, 1977, p. 164).

t is the task of the curriculum to provide these means. Thompson (1984) has
hown that

> " The key feature here is the continual feedback. ... Feedback ... becomes intrinsic to mental actions only when the outcome of the actions is compared with an expectation. ... An ideal curriculum would have the potential of being manipulable by students to the extent that they may generate feedback of whatever degree of subtlety they feel is necessary."

THE COMPUTER: AN APPROPRIATE MEDIUM

f one assumes that the common traits of these theories are basic for the
evelopment of curriculum materials he needs to answer the question whether,
nd how, materials can be designed which incorporate the required features:

(i) representations or models with heuristic efficiency,
(ii) feedback allowing comparison between expected and actual outcome of actions, and
(iii) adaptability of the level of structural complexity and the required level of understanding.

Feature (iii) implies that within the framework of the same materials, activities with considerable level differences should be possible - from unstructured, playful trials to highly structured and complex investigations.

Kaput (1983) has noticed that the availability of computers in the classroom makes an enormous variety of new representations of mathematical ideas accessible to teachers and students. The main assertion of the present paper is that a specific class of these representations have the required features. More than that: features (ii) and (iii) lead directly and almost invariably to the computer as medium for creating suitable representations.

First, as long as the student uses a vocabulary known to the computer, he will receive immediate and reliable feedback. The only other medium which could give a student such feedback is a teacher whose only concern at that time is this particular student - an infrequent situation. What is more important: The computer can be programmed in such a way that the student is in control, whereas teachers have a tendency to take control themselves. In order for the student to remain in control, the computer needs to present the student with an environment on whose objects the student can act according to his own decisions. Such environments have been called "Microworlds" by Papert (1980). A microworld is more open-ended than most instructional software. It will neither give instructions nor problems to the student. It is a structure in which a concept is incorporated. The state of this structure can be changed by a small number of commands available to the student. This open-endedness allows for the flexibility needed to satisfy (iii). At a low level of complexity, a student may be given only the names of some commands and play around with them until he realizes some regularity which he can then proceed to try out more systematically. The student will thus observe and provoke changes in the state of the microworld. His activities will become more and more structured and, as he progresses through the levels, he may acquire the mental structures whose essence is the concept to be learned.

Guidance is essential in this process and may come from the teacher or from written materials accompanying the program (Thompson, 1984). Therefore the

computer does not replace the teacher, nor even the book. It is an additional, suitable learning aid.

As for heuristic efficiency, it is less directly bound to the computer. But heuristically efficient models are as easy or as difficult to create with a computer as without. Moreover, the fact that the student has to act in order for anything at all to happen, enhances the chances of stimulating his reasoning process: If the student wants to achieve a certain result he needs to search for the action giving this result.

As an example, consider the transformation of functions $T: f \to g$, given by $g(ax+b) = cf(x)+d$. T is a simultaneous linear transformation of the independent and the dependent variables. The graph of g is obtained from the graph of f by

(a) an expansion in x-direction by the factor a,
(b) a translation along the x-axis by b,
(c) an expansion in y-direction by the factor c, and
(d) a translation along the y-axis by d.

An appropriate microworld would contain four commands, one for each of the operations a, b, c, and d, and possibly a fifth command for entering a function. The computer screen could present both, the original and the transformed functions in graphical and in analytic form.
A student working at a low level could be limited to two of the four operations; he could also be restricted to integer values of b and d, and to the value 2 for a and c. On the other hand, a student reaching a high level could be expected to have gained some insight into the group of transformations generated by the four operations a, b, c and d ($a \neq 0$, $c \neq 0$). He should be able to compose two general elements of the group and find the inverse of a given element. He should also have understood why (and where) the group is not commutative. On his way from the lower to the higher levels the student needs guidance. Certain options should be available in printed form as part of the microworld; but in the end, it will be the teacher's task to give the advice, to set the options and the limitations and to guide the student according to his personal needs, abilities and preferences.

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CONCLUSION

It has been shown that the computer provides a natural medium for representing mathematical concepts in a way which is consistent with theories of concept learning. It thus appears that microworlds, supported by written instructions and teacher guidance, are a suitable computer environment for concept learning. At the conference, several examples of such environments will be described. Hopefully, some of them will be demonstrated.

It is then natural to hypothesize that these environments may enhance concept learning. Whether or not concept learning actually occurs and whether it occurs more spontaneously, more easily, more naturally than in other situations, remains to be examined. This is a question which only careful research in concept development can answer.

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EFFECT OF COMPUTER GRAPHICS ON IMPROVING ESTIMATES TO AVERAGE SPEED PROBLEMS

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Florida Atlantic University

College students were asked to estimate answers to average speed problems of the following type: What is your average speed for a round trip if you drove at 60 mph in one direction and 10 mph in the other direction? We compared three different instructional procedures to determine which would be most effective in improving estimates. A discovery procedure simulated the round trip by showing a point moving at the two simulated speeds. In a second procedure, students viewed a graph that showed how increasing the initial speed influenced the average speed when the return speed was held constant. In a third procedure, we gave students some problems in which they were shown how to compute the average speed by dividing total distance by total time. The results indicated that the discovery procedure was the most successful in improving students' estimates.

llege students were asked to estimate answers to average speed problems the following type: What is your average speed for a round trip if you ove at 60 mph in one direction and 10 mph in the other direction? We mpared three different instructional procedures to determine which would most effective in improving estimates. All three procedures utilized a crocomputer. In Experiment 1 we contrasted a discovery learning ocedure with a procedure that illustrated the same information in a aph. In Experiment 2 we contrasted the graph procedure with a mputational procedure that showed students how to compute average speed en given the distance of the round trip.

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EXPERIMENT 1

Procedure

Fifty-one undergraduates at Florida Atlantic University participated in the experiment. They were tested individually and were asked to estimate the answers to 10 speed problems, once before the instruction began and once after each of two instructional sessions. All students viewed a graph and participated in a discovery session, although 29 viewed the graph first and 22 did the discovery session first.

The test questions required an estimation of the average speed for each of ten cases which specified different rates of travel for each direction of a round trip. The same questions appeared on each of the three tests. They were preceded by a hint before the pretest that it is necessary to consider how long someone traveled at a specified speed when estimating average speed. Students were informed that if they went on a round trip and averaged 40 mph in one direction and 60 mph in the other direction, their average speed would be closer to 40 mph than to 60 mph because they spent more time traveling at 40 mph.

The graph procedure tested whether students would improve their estimates after viewing a function showing how the average speed of a round trip changes with the initial speed (see Figure 1a). They were told that the example assumes that the return speed is always 10 mph.

The discovery procedure allowed students to obtain feedback through interacting with a computer simulation. The procedure allowed students to view a round trip in which a point moved at the two simulated speeds. A second point simulated the students' estimates and they were informed that both points would complete the trip at the same time if their estimate was correct. Students had three attempts to improve their estimates for each of four examples. The return speed was always 10 mph for each example and the initial speed was either 20, 40, 60, or 80 mph.

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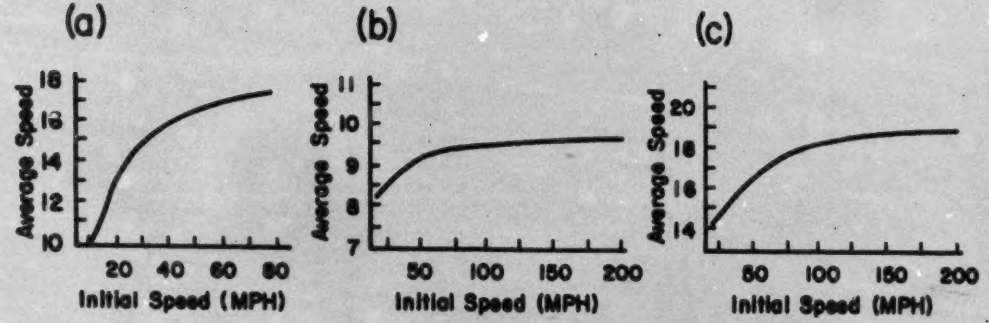

Figure 1. A graph of how the average speed of a round trip varies with the initial speed when the return speed is (a) 10 mph, (b) 5 mph, and (c) 10 mph.

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Results

Our primary measure of the accuracy of an estimate was the absolute deviation between the estimate and the correct answer. A secondary measure of performance was the extent to which students learned the correct concept that the average speed is less than the arithmetical average of the two speeds. Because of space limitations, we will present only the latter data.

Table 1 shows that even with the hint, the majority of responses on the pretest was inconsistent with the correct concept. In the left column are data for students who did the graph session first (Test 1) and the discovery session second (Test 2). The data in the right column are for students who did the discovery session first and the graph session second. The table shows that students reduced the number of inappropriate responses from 58% to 21% following discovery learning. The reduction was only from 55% to 42% after viewing a graph, but was further reduced to 18% following discovery learning. The Test x Sequence interaction was significant, $F(1, 49) = 4.78$, $p < .05$, indicating that the discovery session was more effective.

TABLE 1

PERCENT OF RESPONSES GREATER THAN OR EQUAL TO THE ARITHMETICAL AVERAGE

Experiment 1

	Graph-Discovery	Discovery-Graph
Pretest	55	58
Test 1	42	21
Test 2	18	14

Experiment 2

	Graph-Computation	Computation-Graph
Pretest	50	56
Test 1	37	51
Test 2	32	42

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EXPERIMENT 2

The purpose of Experiment 2 was to evaluate both a modified version of the graph procedure and a different instructional procedure based on computation. We attempted to make the graph procedure more effective by including a second graph and by extending the range of the return speed to make it more obvious that the function is approaching an asymptote. Figures 1(b) and 1(c) show the two graphs used in Experiment 2. In the computation procedure, students were shown how to compute the average speed by dividing total distance by total time.

Procedure

Fifty-six undergraduates at Florida Atlantic University participated in Experiment 2. Except for changes in the two instructional procedures, the experimental design was similar to Experiment 1. All students studied the two graphs and participated in the computational session, although half viewed the graphs first and the other half did the computational session first.

The two graphs were presented sequentially. The first was based on a return speed of 5 mph and the second was based on a return speed of 10 mph. The horizontal axis showed different initial speeds ranging from 25 to 200 mph, as shown in Figures 1(b) and 1(c).

The computational session also included two examples in which students were asked a sequence of questions that required them to calculate speed for different values of distance and time. In one example they were informed that they traveled 50 miles to some location, requiring 10 hours to arrive and 1 hour to return. They were asked to calculate their average speed for each direction of the trip, and for the entire trip by dividing total distance by total time. The computer program provided feedback regarding the correct answers and instructed the students to notice that the average speed for the entire trip (9.1 mph) was much closer to the slower speed

(5 mph) than to the faster speed (50 mph). The second example followed a similar format except that the time of the initial trip was 5 hours instead of 10 hours.

Results

Table 1 shows that the students' performance after the graph sessions in Experiments 1 and 2 were fairly equivalent. From the pretest to Test 1 there was a 13% reduction in both experiments suggesting that adding a second graph and extending the range in Experiment 2 did not seem to enhance the effectiveness of this instructional procedure. Students were even less successful following the computational procedure in which there was only 5% reduction in inappropriate responses. The lack of a Test Sequence interaction, $F(1, 54) < 1$, indicates that the two instructional methods did not differ significantly in their effectiveness.

DISCUSSION

The graphs provided a concrete picture of how the slowest speed constrains the average speed, but they did not allow for the interactive responding that occurred during the computational and discovery sessions. The computational procedure included feedback but could not be applied directly to the estimation test because the questions specified neither distance nor time. The limited amount of learning in the computational session may have been caused by the abstract nature of the task which made it difficult to induce the principle. A key factor for the success of the simulation procedure in Experiment 1 may be that it provided both concrete examples and corrective feedback. A discovery procedure may also help students interpret graphs, particularly if the simulation and graph procedures were combined. In future research we plan to plot the correct answers on a graph after students discover the answers by interacting with a simulation of the event.

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SYMPOSIA

FROM COUNTING TO RECALL OF NUMBER FACTS

Ruth Steinberg - University of Wisconsin
Arthur Baroody - University of Rochester*
Karen Fuson - Northwestern University

In this symposium we examine the transition children make from counting to recall of number facts and the role of derived facts strategies in learning number facts at a recall level. Additionally, we discuss how the number facts are organized in long-term memory and how they are retrieved. We also raise more research questions about the learning of number facts at a recall level and their implications to instruction.

The learning of number facts at a recall level is an important goal of the elementary school curriculum, even in the era of calculators and computers. Extensive research has been conducted in recent years on the solution processes children use to solve addition and subtraction problems. Much of the research has described the development of various counting strategies. Little is known, however, about how children make the transition from counting to recall.

In this symposium we will discuss this transition, the way number facts are organized in long-term memory, and how they are retrieved.

In the first of the following sections, R. Steinberg discusses how derived facts may serve as a transition between counting algorithms and recall of number facts. A. Baroody then discusses how number combinations might be represented in adult long-term memory. In the final section, K. Fuson raises a number of research questions concerning the learning of number combinations.

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DERIVED FACTS STRATEGIES AND THE RECALL OF FACTS

Recent models that describe the learning of number facts at the recall level stress the move from counting to the recall of facts (Ashcraft, 1982; Siegler & Shrager, in press). Derived facts strategies ("thinking strategies") might also play an important role in the transition to recall of facts. A child who uses derived facts strategies derives unknown number facts from a small set of known number facts (e.g., 6 + 7 is 1 more than 6 + 6).

Using derived facts can help children organize the number facts by relating facts to each other. Moreover, children gain understanding and familiarity with the possible sizes of the sums or differences of various number facts. Children who "make sense" of the number combinations do not treat them as random facts but make connections and relationships among them and thus are more likely to generate associations of the number facts. Carpenter and Moser (1983) argued that relating number facts to each other can lend an organizational framework to the set of facts and might provide a logical basis for storing addition and subtraction combinations in long-term memory. This idea is consistent with the current perception of memory. Most researchers believe that knowledge and understanding of a topic help children and adults recall facts (Chase & Simon, 1973; Chi, 1978; Flavell, 1977).

Siegler and Shrager (in press) proposed a model in which children recall number facts when they have a high level of confidence in their ability to do so. Using derived facts might increase children's confidence in their ability to recall facts by enhancing their associations with number facts.

There is some evidence that the use of derived facts strategies might be important for the transition to recall of number facts. Carpenter and Moser (1984) found that a substantial number of children (about 40%) use many derived facts strategies in a transitional period before they learn the facts at a recall level, even though these strategies are not usually taught at school. In another study (Steinberg, 1984), second graders were taught to use derived facts strategies. The children who used many derived facts even before the

study began were the only group of children who mainly used recall of number facts at the end of the study. When these children failed to recall a fact, they used derived facts strategies, rather than counting, as a backup system. Furthermore, a few children who first found 9's combinations by relating them to 10's later recalled more 9's combinations than other facts and used their knowledge of 9's facts as a reference to find other facts.

An important question is how number facts are organized and retrieved from long-term memory. Is the process of retrieving number facts from long-term memory related to the way the facts are learned? Is the quality of long-term memory better after using derived facts? It could be that the use of derived facts strategies eventually becomes automatic and turns into a retrieval process from memory, in which the derived fact is used unconsciously.

One method that has been used to study retrieval of facts is to record individuals' reaction times for solving number combinations. These studies have found that it takes longer for adults and older children to give answers to addition problems with large addends. According to the MIN model (Parkman & Groen, 1971), reaction time is a linear function of the minimum addend. An alternative model predicts reaction time by an exponentially increasing function of the sum of the addends (Aschraft, 1982). A useful way to test these models would be to relate reaction times with verbal reports from people of the actual strategies they used. If the reaction time results still hold among those people who report retrieval from memory, this would suggest that retrieval of facts from long-term memory is not a random access, immediate process in which constant time is spent on every search from memory. Instead, it might be that there is some kind of sequential retrieval process that is related to the way the facts have been thought about before they were stored in long-term memory.

One model that suggests an explanation for the way number facts are stored and retrieved from long-term memory is the stored addition table model, in which a person mentally "looks up" number combinations organized in a form

of an addition table in which retrieval distances from one cell to another are non-linear (Ashcraft, 1982).

If children learn number facts at a recall level after extensive use of derived facts, it is possible that the reaction time of number facts retrieval from long-term memory will be similar for combinations in which a similar derived fact strategy is used (T.P. Carpenter, personal communication, June 1984). For example, if a child uses a "double +1" strategy then all combinations that can be solved in this way (e.g., 6 + 7, 8 + 7, 8 + 9) might be retrieved from memory in a similar way. If quick, unconscious associations are used to retrieve a number fact, it is possible that a person who uses many "doubles + 1" strategies will retrieve the answer by associating the addends with a double close to it. This will result in similar retrieval times for all combinations that were solved by the same derived fact strategy before they were stored in long-term memory. The 9's facts (e.g., 9 + 5, 4 + 9, 9 + 7) might be another group of combinations that might require similar time to retrieve after being solved frequently by relating them to 10.

Derived facts strategies might play an important role in the transition from counting to recall of facts. The use of derived facts might enhance the learning of number facts at a recall level, and might play a role in the way the number facts are represented and retrieved from long-term memory.

REPRESENTATION OF THE BASIC COMBINATIONS

How are the basic whole number combinations represented in long-term memory? Current models of mental arithmetic development posit that counting procedures (slow reconstructive processes) are gradually replaced with rapid fact retrieval (e.g., Ashcraft, 1982; Ilg & Ames, 1951; Resnick & Ford, 1981). It is commonly thought that, through repeated exposure, children just memorize the basic facts (e.g., Siegler & Robinson, 1982). According to the prevalent view, the basic number combinations are organized in an adult's memory as a network of specific numerical associations. Accurate and automatic production of the basic combinations is simply a matter of quickly searching the network and retrieving the appropriate association--that is, looking up the

relevant fact in LTM. In other words, efficient production of the basic combinations by adults is viewed exclusively as a reproductive process. (Slow reconstructive processes may be used by adults when fact recall fails.) However, is the mental representation and efficient production of basic combinations really so simple?

Consider the following analogy. Some idiot savants have the uncanny ability to take a date such as September 17, 1987 and quickly indicate the day of the week the date falls one: Thursday. It was long thought that idiot savants were able to accomplish such feats efficiently because they had a powerful rote memory ability. That is, it was thought they simply memorized calendars in the form of thousands of associations between dates and days or as mental images of the calendars. Thus when given a date, all they had to do was look up in LTM the correct fact (date-day association). However, idiot savants can give correct answers to problems involving calendars they have never been exposed to (e.g., future dates). Thus, it is now believed that idiot savants solve such problems not by looking up stored facts but by quickly computing the answer. What is stored in LTM appears to be an algorithm--a computational procedure--for figuring out date-day connections. In short, it seems that idiot savants use a reconstructive rather than a reproductive process to efficiently generate calendar solutions.

Might the role of reproductive processes in the effecient generation of number combinations by adults by overrated? Might (fast) reconstructive processes play a more important role than previously thought? An alternative view is that adults store only some of the basic combinations in a retrieval network--as specific numberical associations. Many combinations might be accurately and quickly produced from well internalized relationships--from stored rules, algorithms or principles (Baroody, 1983; 1984a). In other words, efficient production of the basic combination might be accounted for by fast reconstructive processes as well as a reproductive process.

By exploiting already internalized relationships, the child eliminates the need to learn and store hundreds of individual numerical associations. Consider some basic subtraction combinations. By abstracting a single rule (N - 0 = N), a child can quickly reconstruct the answers to ten basic facts (1 - 0, 2 - 0 . . . 10 - 0) as well as an infinite number of other problems (e.g., 172 - 0, 1803 - 0, 1,000,004 - 0). The same is true for the N - N = 0 family of facts. In each case, it is unlikely that an adult has stored in LTM an infinite number of associations. In terms of memory space required, the use of internalized relationships is cognitively more economical than storing specific numerical associations.

The alternative model raises the possibility that a member of a combination family might be stored in the form of a specific association and represented by a rule (R. S. Siegler, personal communication, May 18, 1984). For example, 1 + 0 of the zero addition family might be stored as the fact 1 + 0 and represented by the rule N + 0 = N. Less familiar family members such as 0 + 81 or 9077 + 0 would probably be represented solely by the rule N + 0/0 + N = N. Likewise, through repeated exposure, the child can learn the specific numerical association 1 + 1 = 2. This combination could also be represented by the "N after" rule ("adding 1 to an N equals the number after N in the count sequence"). Again less familiar family members such as 1 + 81 or 9077 + 1 would probably be reconstructed by using the N after rule in conjunction with the mental representation of the count sequence.

Such a proposal raises several questions. First, what is the extent of multiple representation? Are all or more of the basic number combinations represented both as facts and by stored rules, heuristics and principles? Though possible, it would not be cognitively economical. It may be that only a few examples of a family are stored both as fact and rule. These examples might serve as focal instances--much as storing the image of, say, a collie might serve as an exemplar of a concept of "dog".

Second, if a combination is represented by both a fact and a rule, which representation is chosen or accessed? This might depend on the associative strength of a combination (cf. Siegler & Robinson, 1982; Siegler & Shrager, in press).

ery familiar combinations with great associative strengths might always be etrieved from a factual representation. The extent to which the factual rep- esentation of less familiar combinations is tapped might depend on the pecific conditional probability of the fact and confidence criterion set y the subject. As noted earlier, unfamiliar combinations would probably be enerated from the representation of the rule.

n conclusion, it does not seem safe to assume--as current models do--that dults represent all the basic number combinations as hundreds of specific umerical associations--as "facts". Current fact retrieval models do not seem efensible on either rational or empirical grounds (see critical reviews of the ental arithmetic literature: Baroody, 1983, 1984a, 1984b). It may be that nternalized relationships and fast reconstructive processes may play a more mportant role in the efficient production of number combinations than is ommonly assumed.

RESEARCH QUESTIONS IN NUMBER FACTS RETRIEVAL

lthough much progress has been made in studying how children learn number acts, many important research questions are still unresolved. Some of these uestions are:

Is fast, automatic retrieval of subtraction facts desirable?
Do children retrieve most subtraction facts from addition facts? If so, how can we best help children to do this? If not, do children learn subtraction facts better if they are organized or practiced randomly?
What amount of speed and accuracy of retrieval of subtraction facts is necessary for good functioning of the division algorithm?
In what other areas of math is fact, accurate retrieval helpful?
Do children reach a complete understanding of the relationship between addition and subtraction or must they recognize it anew in each of its various manifestations (e.g., addition/subtraction with concrete objects, addition/subtraction counting procedures, addition/subtraction facts)?

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* Should instruction in subtraction be considerably delayed?
* What should be the time scheme for te hing various aspects of addition and subtraction? Should each aspect of subtraction (e.g., subtraction facts using the numbers 1-4) immediately follow the corresponding aspects of addition, or should it be considerably delayed?

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AFFECTIVE FACTORS AND MATHEMATICS LEARNING

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ABSTRACT

For many years there has been recognition that personal belief systems which are composed of affective factors influence many components of mathematics learning. In the past, much research dealing with affective factors has been inadequately conceptualized. Poorly defined variables have been studied using inadequate measurement procedures in atheoretical settings. Most studies have reported simple correlations between these poorly defined and measured variables and mathematics learning determined by standardized tests. In fact, much of this research has not recognized the complexity of affective factors, mathematics learning, or the relationship between the two.

In the past, research about affective factors in mathematics learning has been weakened by lack of clear definitions for constructs, failure to base studies on earlier research, and ignorance of the related psychological literature. In order to build a more extensive body of knowledge about affective factors in mathematics education, systematic, theory- or model-based research is needed using a variety of methodological approaches. The relationships between affective factors and mathematics learning are complex. It is no longer sufficient to simply examine correlations between affective factors and mathematics achievement. Future research should pursue causal and interactive relationships between affective factors and mathematics achievement.

Study of causal and interactive relationships between affective factors and mathematics seems particularly important in problem solving. Although most theoretical models of problem-solving performance do not

include affective factors, there is no reason why they cannot, and in fact, recent work in cognitive science is beginning to place more emphasis on affective issues. As researchers explore the role of metacognitive processes in problem solving, for example, they also need to clarify the influence of affective factors such as anxiety and willingness to persist. The metacognitive processes that problem solvers need seem very closely tied to affective concerns: How does the problem solver decide when to give up on one strategy and try another? How does one prevent the tension that builds during a problem-solving episode from raising anxieties that promote inflexibility and use up the short-term memory that is needed to try new approaches? What is the influence of anxiety or confidence on retrieval of information from long-term memory that is needed to solve a problem?

Affective factors influence problem solvers at both the conscious and unconscious levels. The role of unconscious processing of information has long been of special interest of mathematicians, but little has been done to investigate the relationship between anxiety, for example, and the role of the unconscious in problem solving.

Research on affective factors in mathematics learning has tended to focus on negative emotions like anxiety rather than positive ones like confidence. Yet the central feature of problem-solving instruction needs to be the development of strategies to help motivate students to succeed in problem solving. Researchers need to investigate the cycle of tension and relaxation that brings satisfaction to problem solvers and motivates them to attempt new and creative solutions to problems.

Many important issues dealing with research on affective factors can be identified. However, what is badly needed at the present time is theoretical models that illustrate the relationship between affect, learning, and teaching. Such models must use well-defined constructs, clearly demonstrate how such constructs effect learning and hypothesize how the constructs are developed. A research agenda based on such models would be profitable.

SEX-RELATED DIFFERENCES IN MATHEMATICS LEARNING

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Margaret R. Meyer, University of Wisconsin--Madison
Lindsay Anne Tartre, National College of Education
Jacquelynne Eccles, University of Michigan

For the last decade, there has been much research done in the area of sex-related differences in mathematics learning, achievement, and participation. While the research in this area has sparked some controversy, the one irrefutable conclusion of the scholarly work done thus far, is that the issue of sex-related differences in mathematics is indeed a complex and multi-faceted one. In this symposium, we propose to discuss three facets of the problem. We will consider the influence of cognitive variables, affective variables, and classroom variables.

In discussing cognitive variables, Lindsay A. Tartre will focus more specifically on spatial skills. She will explore the relationship between these skills and mathematics learning. How spatial skills have become involved in the discussion of differential mathematics learning of males and females will be addressed.

Margaret R. Meyer will consider the influence of certain affective variables. One affective variable receiving attention recently has been that of attributional style. Learned helplessness is one such style that is perhaps related to differences in mathematics achievement and participation for boys and girls.

Lastly, the facet of classroom variables will be discussed. Mary S. Koehler will look at those aspects of classroom life that are often different for boys and girls. Teacher-pupil interaction is one such aspect. She will also look closely at the classroom processes that are often associated with effective mathematics teaching. These two areas of research will then be related in an effort to further understand the underachievement of females in mathematics learning.

Serving as reactor to the panel will be Jacquelynne Eccles. She will give the perspective of a psychologist to this problem in mathematics education.

THE ROLE OF COMPUTER PROGRAMMING IN THE DEVELOPMENT OF MATHEMATICAL CONCEPTS

Patricia S. Wilson, organizer
The Ohio State University

Alfinio Flores
Centro de Investigacion en Matematicas

Thomas Kieren
University of Alberta

Mary Kim Prichard
University of North Carolina at Charlotte

Richard J. Shumway
The Ohio State University

Sigrid Wagner
University of Georgia

The focus of the symposium will be on the role of computer programming in the development of mathematical concepts. The ideas explored will include variables, iteration, recursion, use of language, examples and graphics. The symposium was organized to accomplish the following purposes:

1. to share current research,
2. to relate field observations of computer programming by young children to research in concept development, and
3. to encourage coordinated research that will explore the role of computer programming in concept development.

Findings based on observations of programming by young children (K-6) will be reported. Reactors will relate their research in concept development to the reported field experiences. The audience will be encouraged to share field based experiences, research or proposed studies focused on the role of computer programming in concept development.

FIELD EXPERIENCES

Exploring Computer Programming in BASIC with Kindergarteners
Richard J. Shumway
The Ohio State University

The purpose was to identify mathematics and computer skills which could be learned by kindergartners working with simple three- and four-line programs in BASIC. The sample was 26 kindergartners in a single classroom in an integrated, urban, elementary school. The treatment consisted of five 30-minute lessons involving counting and four lessons involving graphing. Concepts encountered were: variables, counting, addition, repeated addition, FOR-NEXT loops, PRINT, Cartesian coordinates, PLOT, the use of variables with ordered pairs, and the relationship between numbers, variables, and geometry. We were impressed with those things which all students learned to do. The successes of some students suggest with more time and opportunity, kindergartners may learn significant, new mathematics by programming in BASIC.

Programming in LOGO with Intermediate Students
Patricia S. Wilson
The Ohio State University

Students in grades 4, 5, and 6 were introduced to LOGO commands that make the turtle move and repeat sequences of moves. They also were taught how to create a program and how to modify a given program. Students worked through a series of activities that lead to building regular polygons using the REPEAT command and manipulating variables in a given program. Students were observed individually and were allowed to question and interact with the instructor. The use of the LOGO turtle graphics seemed to encourage exploration and the use of both teacher and student generated examples. The concepts of iteration and variables were explored. The students were able to understand the iterative process easily and some students were able to manipulate variables in order to accomplish their tasks of building specific polygons. The ease of having the computer draw the polygons encouraged students to try their ideas and experience the power of manipulating variables.

Thomas Kieren
University of Alberta

Commehts will be based on recent field study, that explored the relationship

of developmental levels and the ability to understand LOGO programming processes. The focus will be on the use of language.

REACTIONS

Alfinio Flores
Centro de Investigacion en Matematicas
Guanajuato, Mexico

Reactions will be based on experiences using computers to teach mathematical concepts in the United States and Mexico. Alfinio Flores has worked with both students and teachers and will be able to contrast approaches in the United States and Mexico.

Experienceas sobre el uso de la computadora para el desarrollo de conceptos matemáticos con niños en México serán contrastadas con experiencias presentadas en el simposio. Investigación sobre el desarrollo de conceptos matemáticos con maestros de secundaria y primaria en Guanajuato será descrita.

Mary Kim Prichard
University of North Carolina At Charlotte

Reactions will be based on research investigating the use of the computer as an instructional device for teaching algebra. Kim Prichard has studied the role of computer programming in the development of the concept of mathematical iteration.

Sigrid Wagner
University of Georgia

Reactions will be based on extensive research investigating students' understanding of variables and literal symbols. In addition, Sigrid Wagner offers a perspective based on her studies in cognitive development and logical reasoning as well as language and graphics as representational systems.

SYMPOSIUM PROPOSAL

Merlyn J. Behr

Northern Illinois University

This symposium will be organized as a work-group for the purpose of exploring the current research issues in rational number. Three short presentations (10-15 minutes) will be made for the purpose of giving an overview of the questions and hypotheses arising in three areas of rational number research. The purpose of these presentations will be to stimulate discussion concerning the current state of and future directions for research related to those questions and hypotheses. The objective of the symposium would be to bring together current issues concerning rational number research and point directions for future research. The symposium would differ from past PME-NA sessions in that presentations would not be presented for publication in the Proceedings but would be for generation of activity for a work-group.

The symposium is proposed for two hours and would be organized with a panel consisting of Thomas Kieren and Yvonne Pothier, James Hiebert, and Diana Wearne, Thomas Post, and Merlyn Behr. One person from each to the person-pairs would make a 10-15 minute presentation. The general topics of the presentations would be:

Kieren and Pothier -- Research on partitioning mechanisms and their implications for rational number development.

Hiebert and Wearne -- Research in decimal computation, hypotheses about misconception and research on remediation.

Behr and Post -- On the question of the size or quantitative concept of rational number or what does it mean to think of 3/5 as an entity rather than as a pair.

THE TEACHING AND LEARNING OF GEOMETRY

Alan H. Schoenfeld

The University of Rochester

The purpose of this symposium is to bring together four researchers who have focused on geometry, for what might be called a "half presentation and half working session." Alan Hoffer (University of Oregon), Rosamond Tischler (Brooklyn College), and Sharon Senk (Syracuse University) have all been involved in aspects of Van Hiele-based research; Alan Schoenfeld (University of Rochester) has explored students beliefs about geometry and their effects on cognitive processes. Among the issues to be addressed are (i) the "state of the art" in our understanding of students abilities to deal with geometry, (ii) the state of Van Hiele-based research at present, (iii) the effects of the ways that geometry is taught on students attitudes toward and performance in geometry, and (iv) future directions for research in the area.

The symposium will last for three hours. Each of the four participants will take up to a half hour to present an aspect on his or her own recent work in geometry. Hoffer's work includes broad syntheses of Van Hiele-based research and detailed interviews with students regarding their understanding of geometric concepts. Tischler's work includes an analysis of geometry textbooks, and the kinds of things that students can get out of them. Senk has examined (the absence of) sex differences in proof construction, and affective/belief aspects of performance in geometry. Schoenfeld has looked for the roots of students' attitudes towards geometry in the ways that they are taught the subject.

The purpose of these presentations is to establish a factual basis for dialogue and interactions regarding learning and understanding

in geometry. In the half hour following the individual presentations, the panelists will react to aspects of each other's talks, look for areas of commonality and difference, try to examine/resolve areas of conflict, and point to areas where further inquiry might be profitable. (That is, they will argue among themselves about what has been presented.) For the final half hour, the audience will be invited to join the fray.

A CONSTRUCTIVIST APPROACH TO MATHEMATICS INSTRUCTION

Jere Confrey

Mount Holyoke College

An organizing theme of the 1983 Annual Meeting of PME-NA was the philosophy of constructivism, its meaning and implications for the learning of mathematics. These discussions and presentations focused on the meaning of knowledge, on the importance of learner's activity, and on the inherent ambiguity in communication among teachers and students in classrooms. What failed to emerge from these discussions were models of curricular development and instruction which are compatible with these epistemological perspectives. We believe that the presentation of full blown models of instruction is premature; however, we also believe that our experience over three years designing and conducting two mathematics programs implementing a constructivist's perspective allows us to begin to identify some components of such a model of curricular development and instruction in a reasonable way. In this symposium, we propose to present two conceptual frameworks for examining curricular and instructional practices, and through the presentation of videotapes of our instructors discuss, debate and revise together with the audience the utility of such frameworks in relation to practice. Our final contributor will lead a discussion focused on the possibilities of a constructivist theory of instruction. It is our intent that through such a group activity, we can progress towards a more developed model while exemplifying constructivist commitments within a research conference.

In the first presentation, Confrey proposes some components of constructivist instruction based on the four tenets of constructivism described in Confrey's 1983 PME-NA paper and from the development of Summerth program at Mount Holyoke College. These components suggest that a constructivist view of curriculum and instruction begins with listening to

students to find and explore existing conceptions, designing problems to encourage flexibility, promoting reflection through work in small groups and pairs, developing more adequate problem-solving strategies and considering such affective and sociological issues as risk, confidence, and views of authority. By analyzing a videotape of one of the SummerMath instructors teaching and reflecting on that instruction, the group attending the session can reflect on the meaning, utility, and validity of such components.

In the second presentation, Lochhead describes the instructional method of a precalculus program at the University of Massachusetts. In an attempt to differentiate constructivist instruction from more traditional approaches, a simple, somewhat behaviorist, observational scheme will be suggested in which teacher behaviors are divided into the following categories: observing or listening to students; asking questions, open-ended and specific; providing feedback to students, showing or telling, directing student attention, and repeating or rephrasing the students' ideas. The above scheme makes no attempt to capture such important variables as the quality of a teacher's activities or the degree of emotional support offered to the student. Using such a scheme, an analysis of a videotape of instructors in the program will be presented. Together with the audience, he will discuss the effectiveness of such a scheme for sensitizing prospective teaching assistants for the course, and the dimension of the underlying theory of instruction will be explored.

In the final presentation, Konold explores the possibilities for a constructivist theory of instruction. Drawing largely on issues in the two previous presentations, the attempt is made to derive a set of general instructional goals, cognitive, social and affective, as well as a set of corresponding and coherent instructional methods. The adequacy and uniqueness of this instructional theory is tested by contrasting it with several contemporary instructional approaches.

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